Approximation-III

Pricing Method, LP and PTAS

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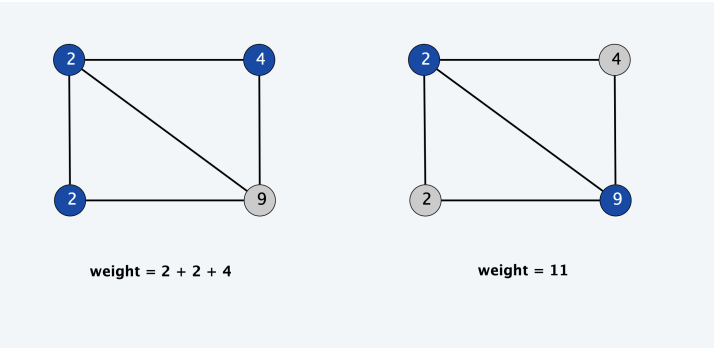
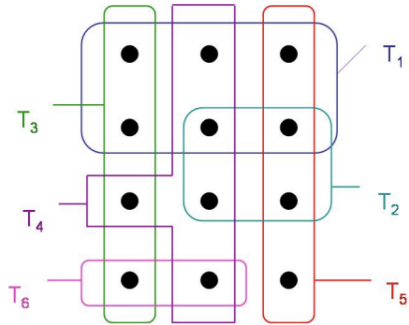
Weighted Vertex Cover



Input: A graph G and $w\colon V(G) \to \mathbb{R}_{\geq 0}$
Output: A minimum weight vertex cover of G.

Image credit: Algorithm Design by Kleinberg and Tardos

Weighted Set Cover



Input: A universe U, family $\mathcal{F}$ of subsets of U, and $w\colon \mathcal{F} \to \mathbb{R}_{\geq 0}$
Output: A minimum weight subfamily of $\mathcal{F}$ covering U.

Image credit: Lecture notes by Amit Chakrabarti

Recall

- 2-approximation for (Unweighted) Vertex Cover.
- $O(\log d^*)$-approximation for Weighted Set Cover.
- $O(\log \Delta)$-approximation for Weighted Vertex Cover

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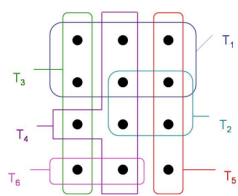
We used pricing method to analyse the algorithm for Weighted Set Cover.

Excursion Problem modeled as Weighted Set Cover

- Universe $U =$ we.
- Sets in $\mathcal{F}$ are groups who like to go together in a car/bus/train etc.
- Weight of the sets corresponds to the cost of the travel.

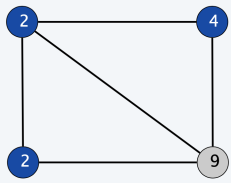


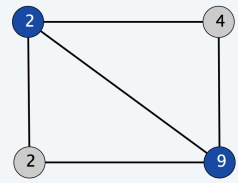
Pricing Method

with fairness and cheating

Example: 2-approximation algorithm for Vertex Cover.

Weighted Vertex Cover





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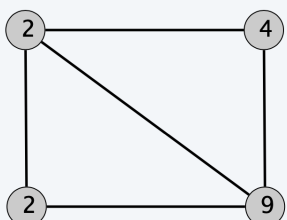

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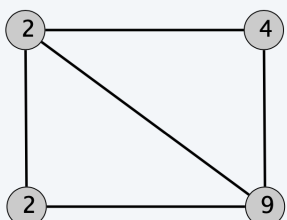

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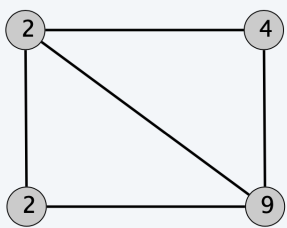

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3: For a vertex x, $\sum_{e=(x,z)} p_e \leq w(x)$

Fairness Lemma

Let $\{p_e \ : \ e \in E(G)\}$ be fair prices. Then for any vertex cover S, $\sum_e p_e \leq w(S)$.

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Corollary: $\sum_e p_e \leq \mathsf{opt}$

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Compute prices p_e and find vertex cover simultaneously.

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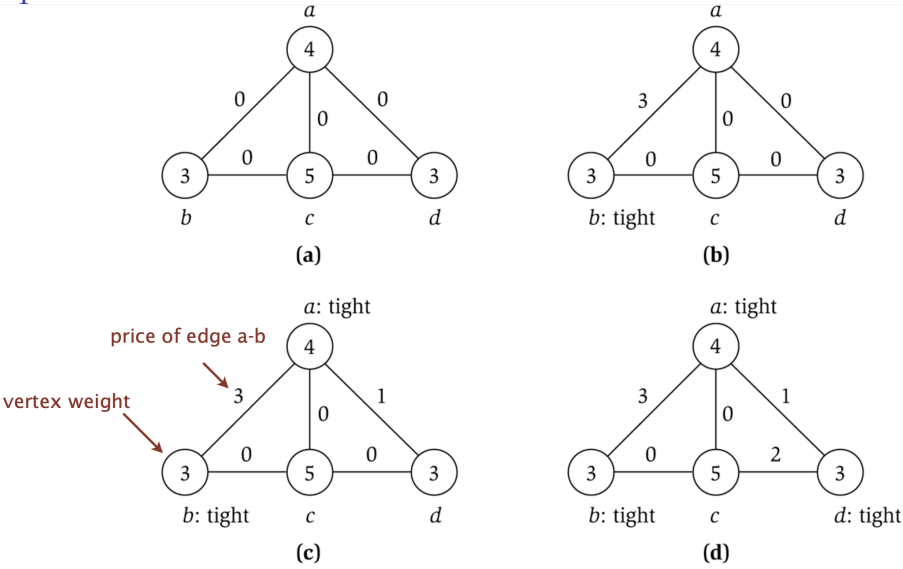
Initially $p_e = 0$ for all e

While(there is an edge $e = (x, y)$ such that both x and y are not tight)
    Select such an edge e and increase p_e until x is tight or y is tight.
EndWhile

Return the set S of tight vertices,.

Example


a
4
0
0
0
0
0
3
5
3
b
c
d
(a)
a
4
3
0
0
0
0
3
5
3
b: tight
c
d
(b)
a: tight
price of edge a-b
vertex weight
4
3
1
0
0
0
3
5
3
b: tight
c
d
(c)
a: tight
4
3
1
0
0
2
3
5
3
b: tight
c
d: tight
(d)

Analysis

Claim 1

Output S is a vertex cover.

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Proof: For each edge e, one its endpoint is tight at the end of the while loop.

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Claim 2

S is a 2-approximate solution

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$$w(S) = \sum_{x \in S} w(x)$$

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So far..

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- $O(\log d^\star)$-approximation for Weighted Set Cover.
- 2-approximation for Weighted Set Cover.
- So, Weighted Set Cover instances derived from Weighted Set Cover has a 2-approximation algorithm.

Linear Programming

Weighted Set Cover: ILP

Variable x_S for each set $S \in \mathcal{F}$.

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Feasible solutions of ILP is in one-one correspondence with the feasible solutions of Weighted Set Cover.

Weighted Set Cover: LP relaxation

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~~Feasible solutions of LP is in one-one correspondence with the feasible solutions of Weighted Set Cover?~~

Linear Programming

Simplex algorithm. [Dantzig 1947] Can solve LP in practice.

Ellipsoid algorithm. [Khachiyan 1979] Can solve LP in poly-time.

Interior point algorithms. [Karmarkar 1984, Renegar 1988, ...] Can solve LP both in polynomial time and in practice.

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- Solve LP and get an optimum solution $x^\star$

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Q is a set cover.

$w(Q) \leq f \cdot \sum_{S \in \mathcal{F}} w(S) \cdot x^\star_S \leq f \cdot \mathsf{opt}$

Polynomial Time Approximation Scheme

PTAS

An algorithm that takes an instance of an optimization problem and a parameter $\epsilon > 0$ and produces $(1 + \epsilon)$-approximate solution.

Knapsack problem

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- Given n objects and a knapsack of capacity W.
- Item i has value $v_i > 0$ and weighs $w_i > 0$.
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item	value	weight
1	1	1
2	6	2
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original instance (W = 11)

Items $\{3, 4\}$ has value 40.

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- It is NP-Complete

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$$T[i,v] = \begin{cases} 0 & \text{if } v \leq 0 \\ \infty & \text{if } i = 0 \text{ and } v > 0 \\ \min\{T[i-1,v], w_i + T[i-1, v-v_i]\} & \text{otherwise} \end{cases}$$

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Polynomail time when the values are small integers.

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- Run the above algorithm with these small values.
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original instance (W = 11)

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Let $\bar{v}_i = \widehat{v} \cdot \theta = \lceil \frac{v_i}{\theta} \rceil \cdot \theta$

Observation: Optimal solutions to problem with v are equivalent to optimal solutions to problem with $\widehat{v}$.

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Observation: Optimal solutions to problem with v are equivalent to optimal solutions to problem with $\widehat{v}$.

Intuition: $\bar{v}$ close to v. So optimal solution using $\bar{v}$ is nearly optimal for v.

Analysis: approximation ratio

- Let S be the output of our algorithm and $S^\star$ be an optimum solution.
- To prove: $(1+\epsilon)\sum_{i\in S} v_i \geq \sum_{i\in S^\star} v_i$.

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&\leq \sum_{i\in S} \bar{v}_i \\
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&\leq n\theta + \sum_{i\in S} v_i
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Analysis: approximation ratio

- Let S be the output of our algorithm and $S^\star$ be an optimum solution.
- To prove: $(1+\epsilon)\sum_{i\in S} v_i \geq \sum_{i\in S^\star} v_i$.

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&\leq \sum_{i\in S} \bar{v}_i \\
&\leq \sum_{i\in S} (v_i + \theta) \\
&\leq n\theta + \sum_{i\in S} v_i \\
&\leq \frac{1}{2}\epsilon v_{max} + \sum_{i\in S} v_i
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Summary

- Pricing Methods
- Linear Programming and rounding
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Thank You.