Approximation-I

Pricing Method and LP

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Outline

- Approximation algorithms.
- Greedy algorithms. Analysis using pricing method.
- Linear Programming and rounding.

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Problems:

- (Weighted) Vertex Cover
- (Weighted) Set Cover

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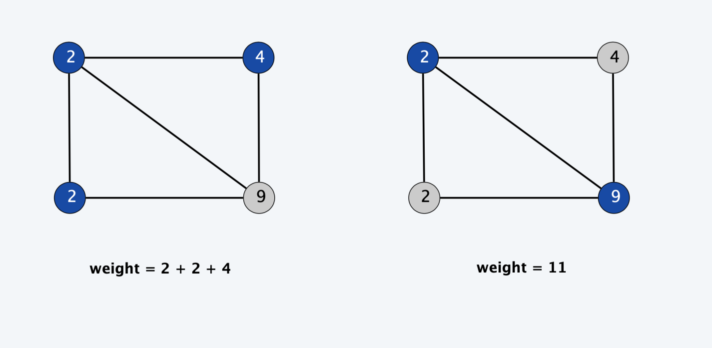
α-approximation algorithm, where $\alpha > 1$

Finds solution in polynomial time that is within ratio α of optimum.

Weighted Vertex Cover



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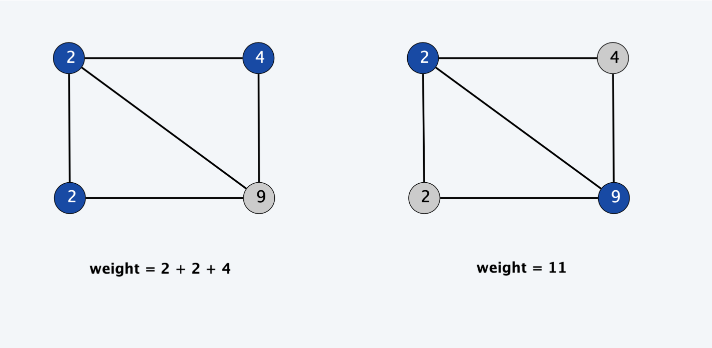
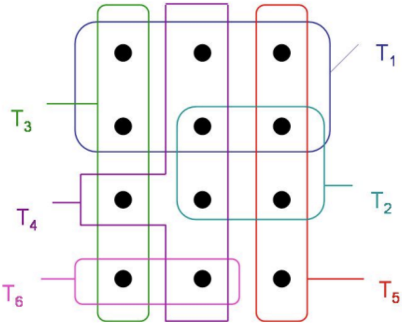


Input: A graph G and $w : V(G) \to \mathbb{R}_{\geq 0}$
Output: A minimum weight vertex cover of G.

Image credit: Algorithm Design by Kleinberg and Tardos

Set Cover



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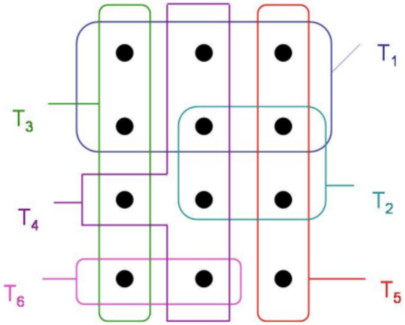

Input: A universe U, family $\mathcal{F}$ of subsets of U, and $w\colon \mathcal{F} \to \mathbb{R}_{\geq 0}$
Output: A minimum weight subfamily of $\mathcal{F}$ covering U.

Image credit: Lecture notes by Amit Chakrabarti

Warm up: Greedy Vertex Cover

Input: A graph G
Output: A minimum cardinality vertex cover of G.

Start with $S := \emptyset$
While $E(G) \neq \emptyset$
    Select an edge $e = (x, y)$.
    $S := S \cup \{x, y\}$.
    $G := G - \{x, y\}$
EndWhile
Return S

Tight analysis?

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Khoot–Regev 2006

If Unique Games Conjecture is true, then no $(2 - \epsilon)$-approximation algorithm for Vertex Cover

Pricing Method

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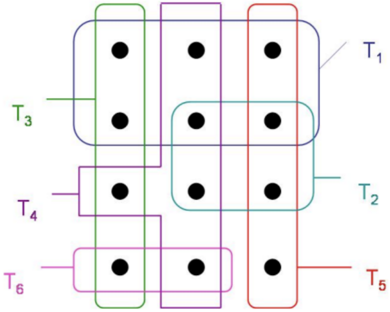

How to solve?

Greedy Algorithm

Start with $R := U$ and no sets selected

While $R \neq \emptyset$

    Select S_i that minimizes $\frac{w(S_i)}{|S_i \cap R|}$

    Each $x \in S_i \cap R$ pay a cost $c_x = \frac{w(S_i)}{|S_i \cap R|}$

    $R := R \setminus S_i$

EndWhile

Return the selected sets

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- Then, $\sum_{S \in \mathcal{C}} w(S) = \sum_{x \in U} c_x$.
- Now, for each $S \in \mathcal{F}$, we upper bound $\sum_{y \in S} c_y$.

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&= H(d^\star) \cdot \mathsf{opt} \\
&\leq 2(\log d^\star) \cdot \mathsf{opt}
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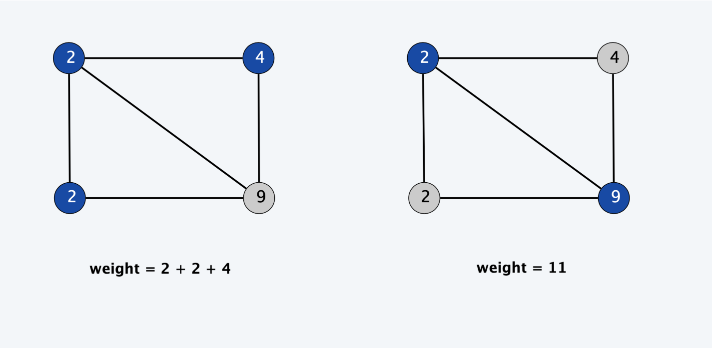

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Next: 2-approximation algorithm for Vertex Cover.