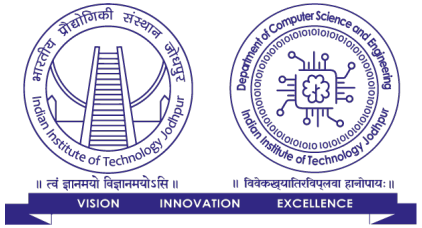


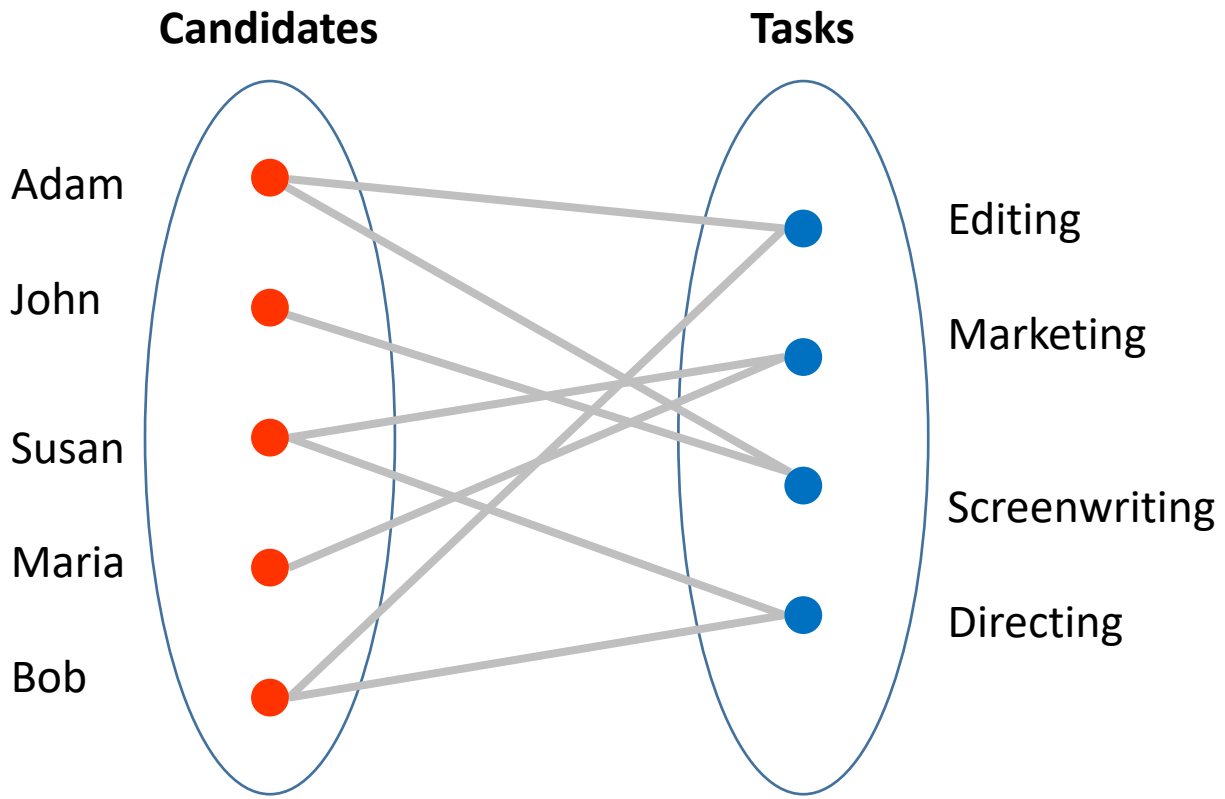
Matching Theory and Vertex Cover

Kshitij Gajjar

Assistant Professor, Dept. of CSE,
Indian Institute of Technology Jodhpur

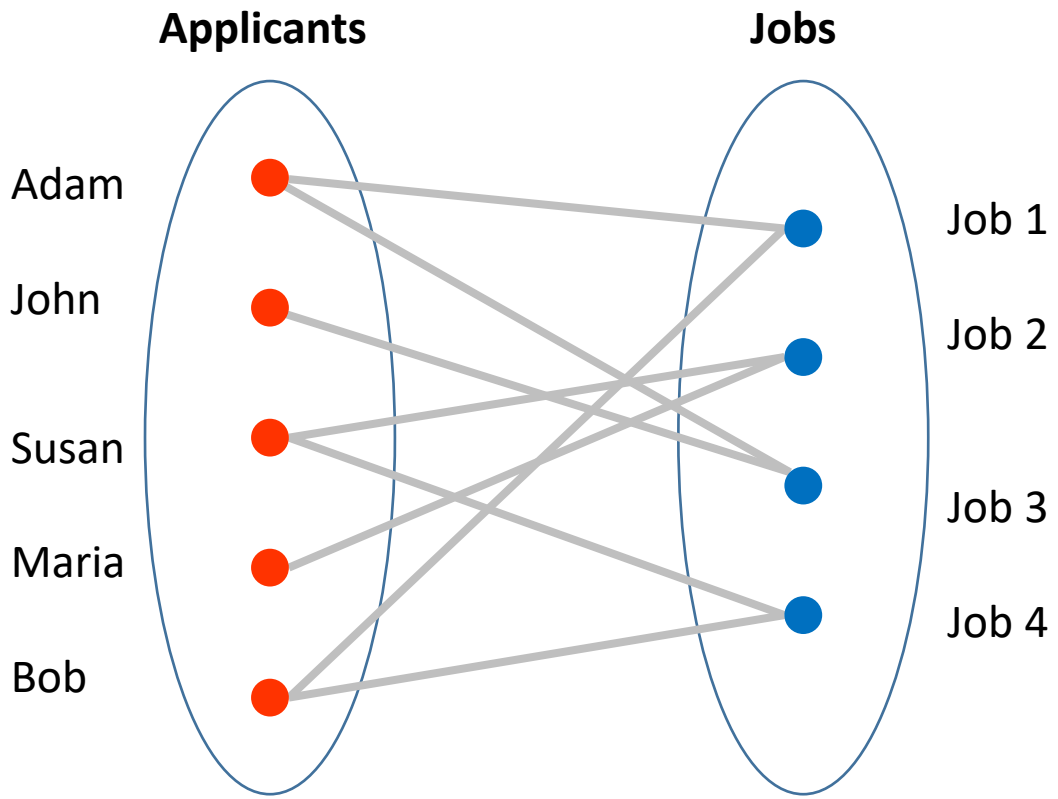
Why Matchings?

- Suppose a group of people are making a skit/drama/play.
- Each person can only do a certain set of tasks.
- How do you assign the tasks?



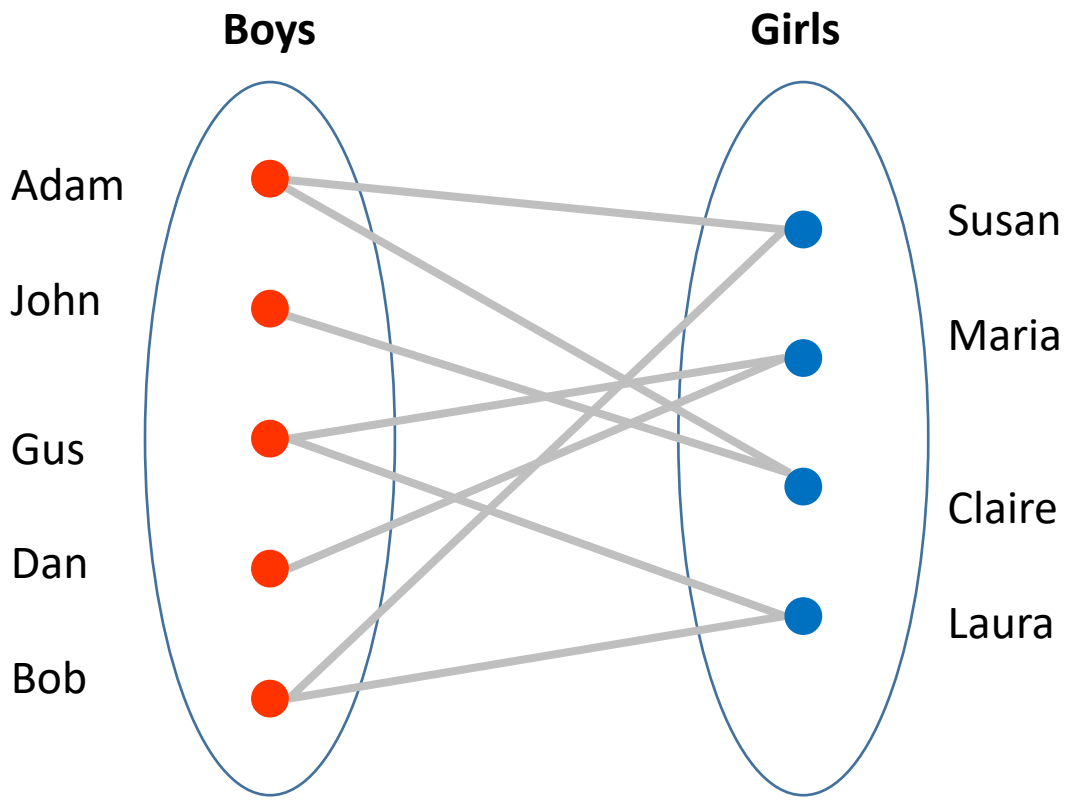
Why Matchings?

- More generally, job applicants can be assigned to jobs.



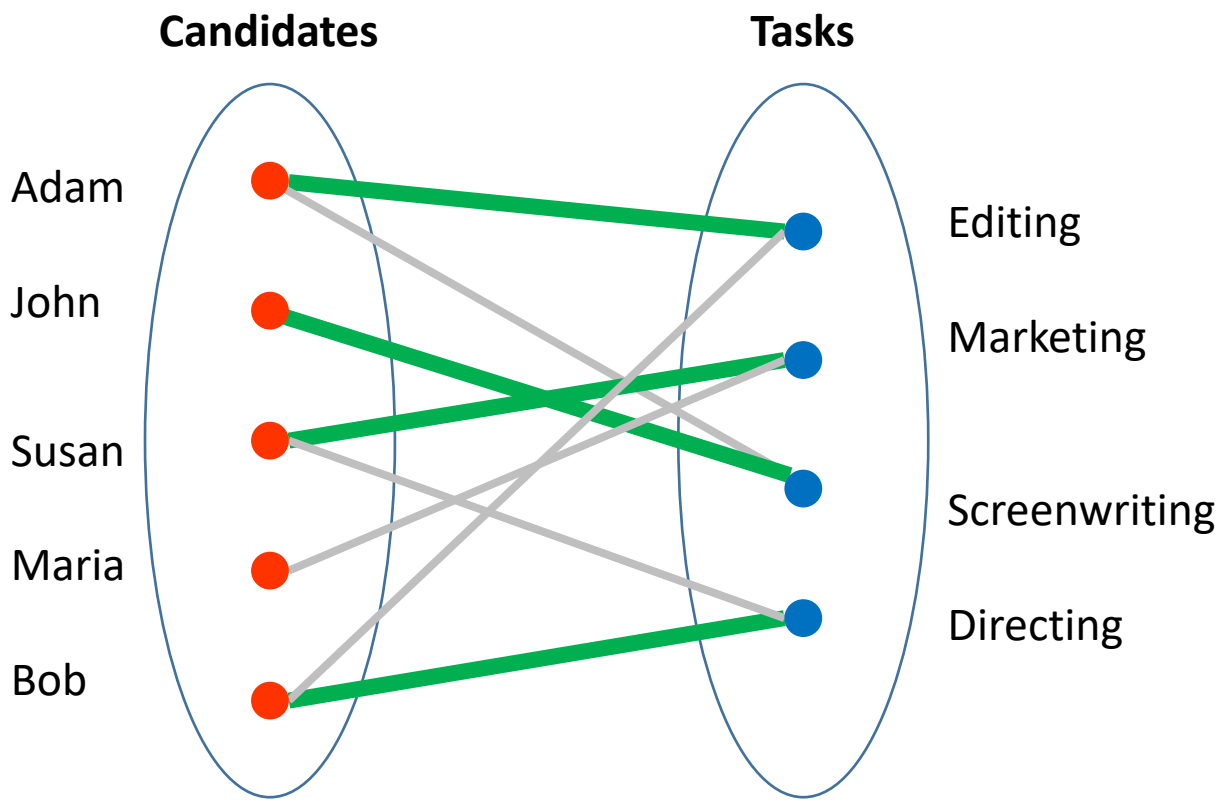
Why Matchings?

- More specifically, partners can be assigned for marriages.
- More on this later…



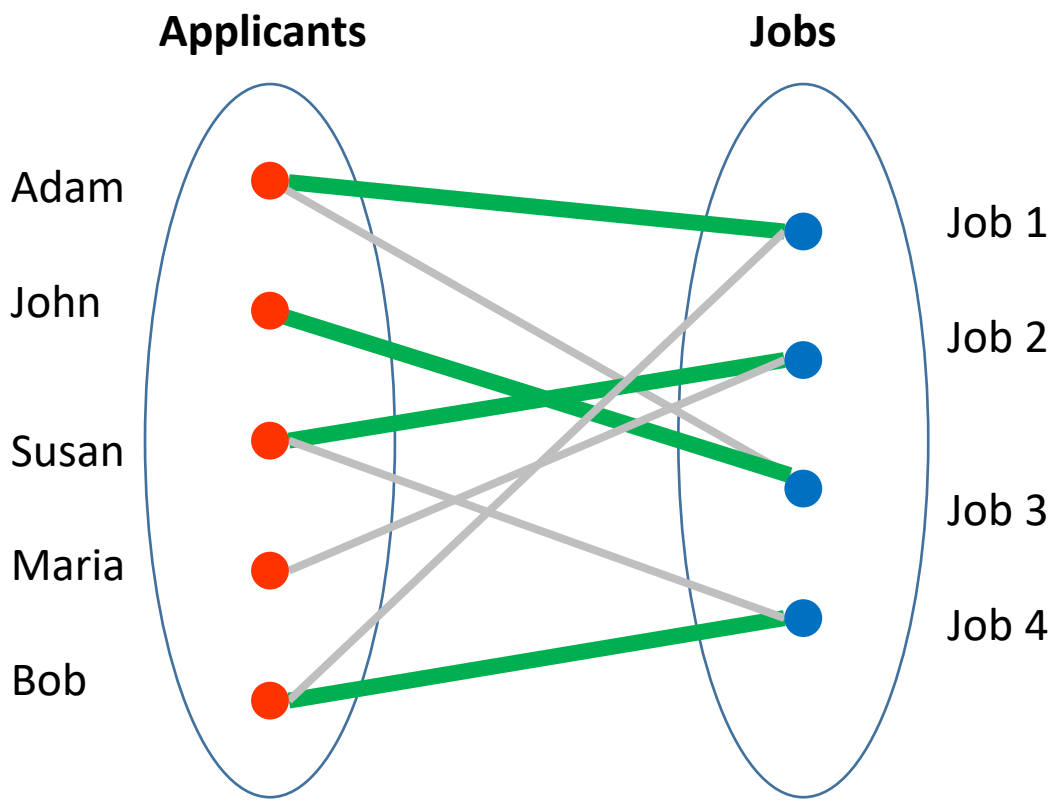
Maximum Matching

- Adam: Editing
- John: Screenwriting
- Susan: Marketing
- Bob: Directing



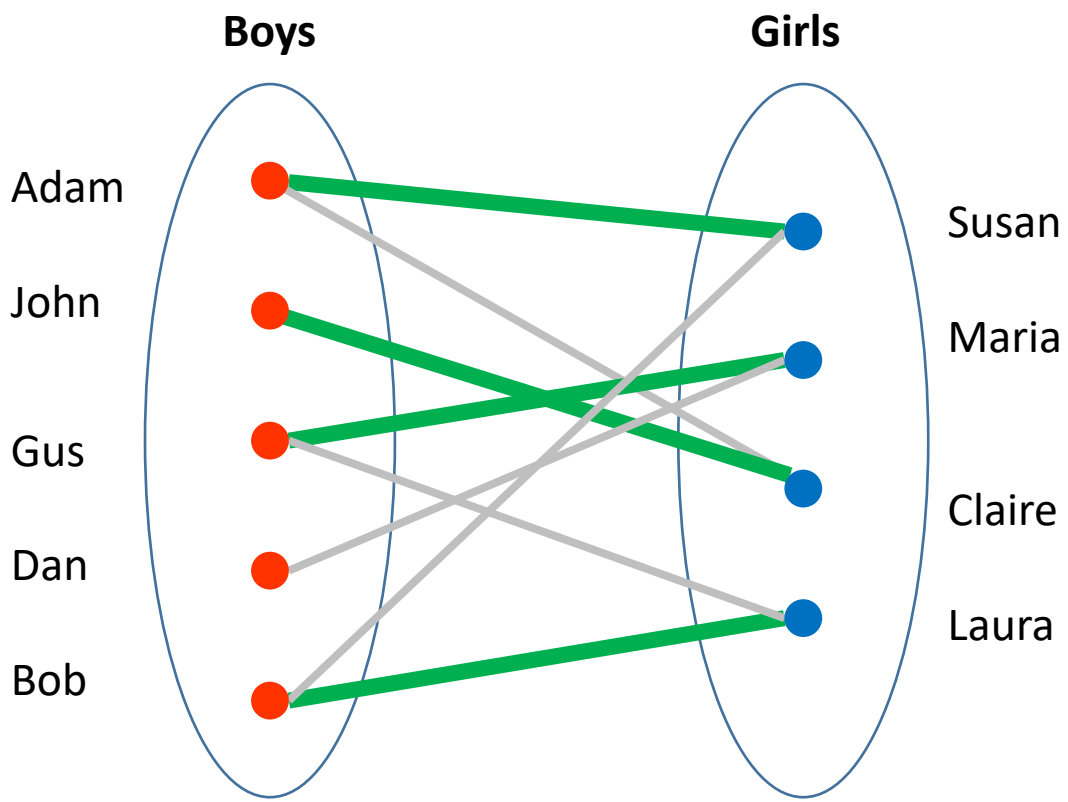
Maximum Matching

- Adam: Job 1
- John: Job 3
- Susan: Job 2
- Bob: Job 4



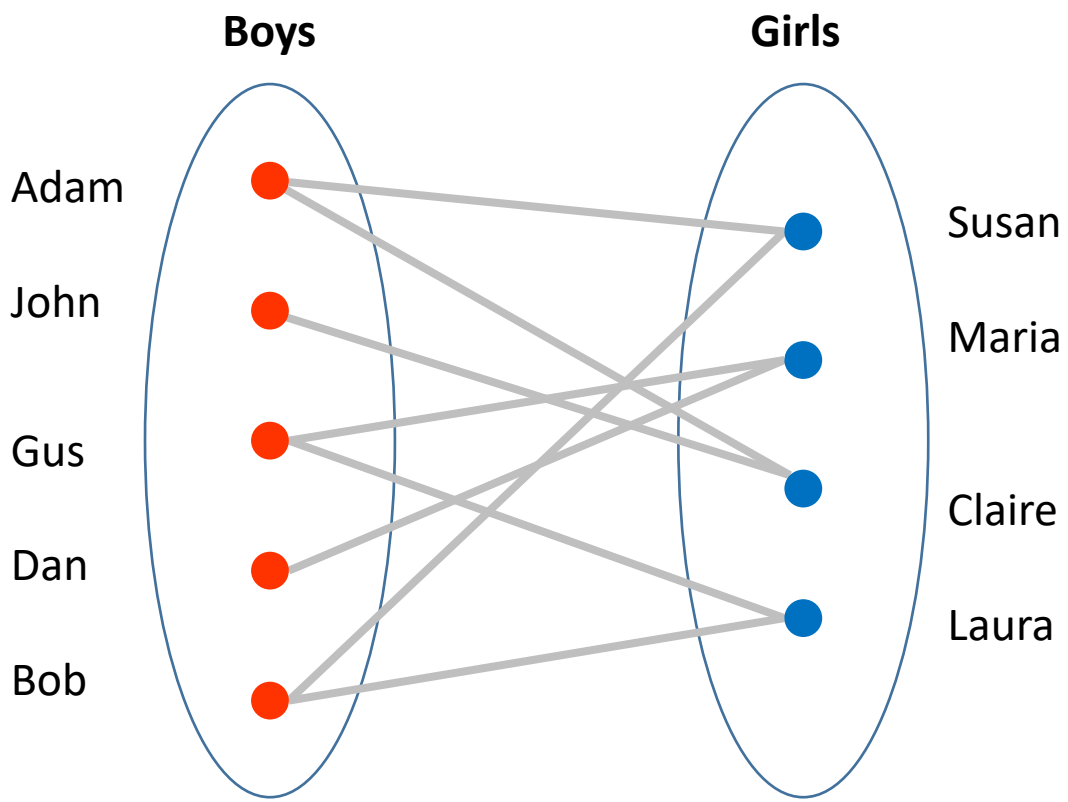
Maximum Matching

- Adam: Susan
- John: Claire
- Gus: Maria
- Bob: Laura



How to Find a Maximum Bipartite Matching?

- Part of matching
- Not part of matching



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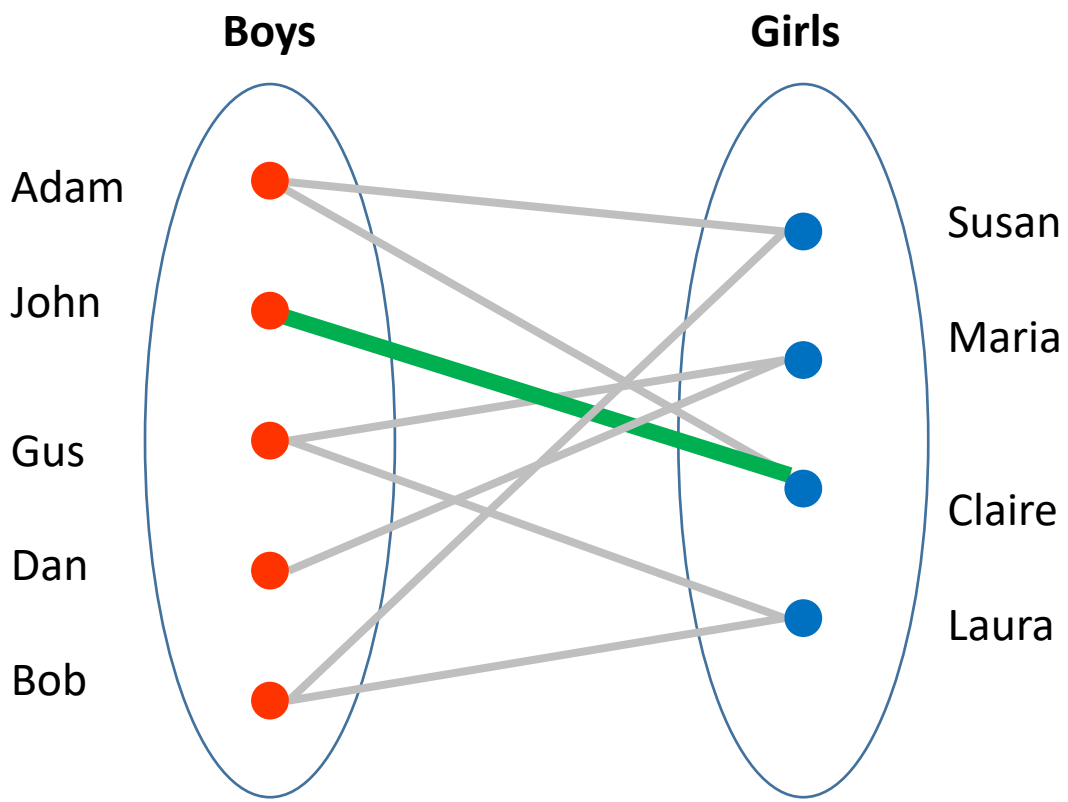

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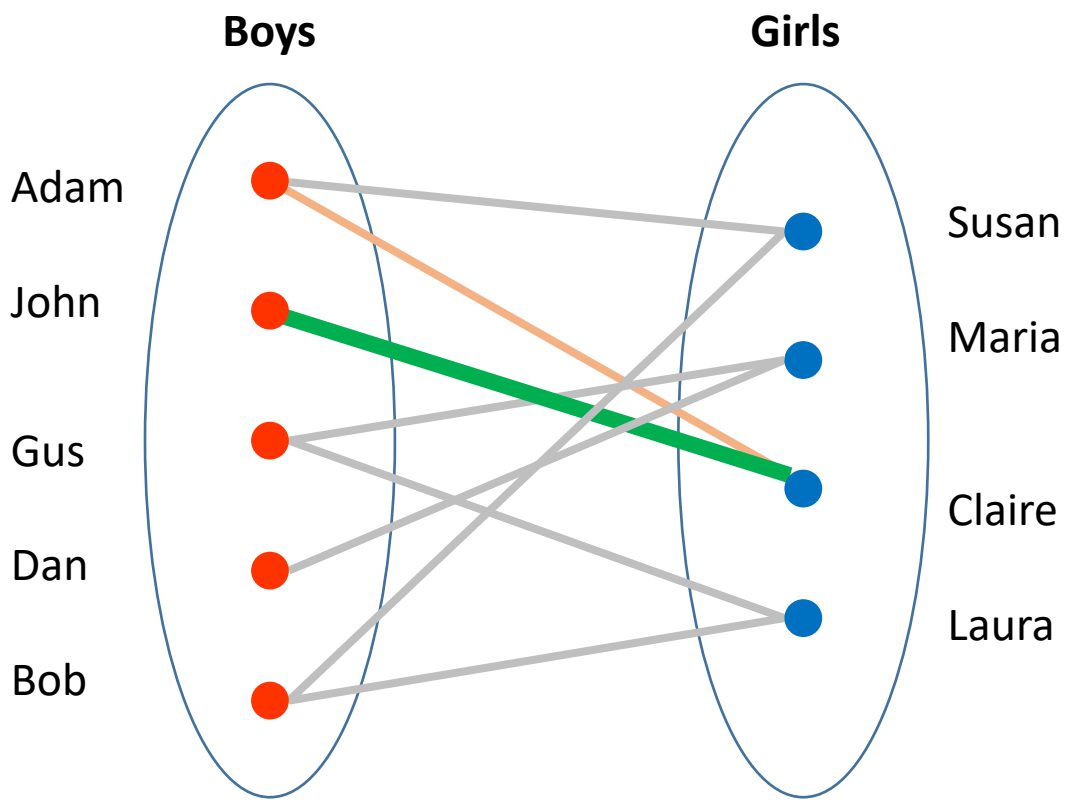

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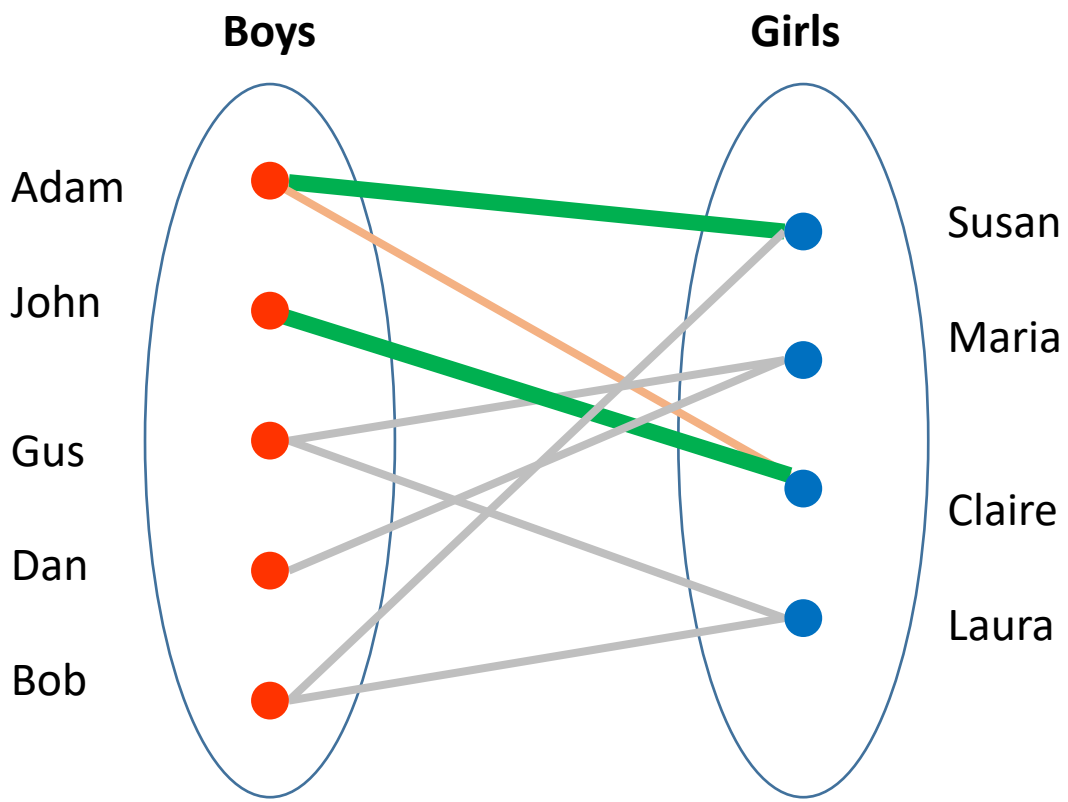

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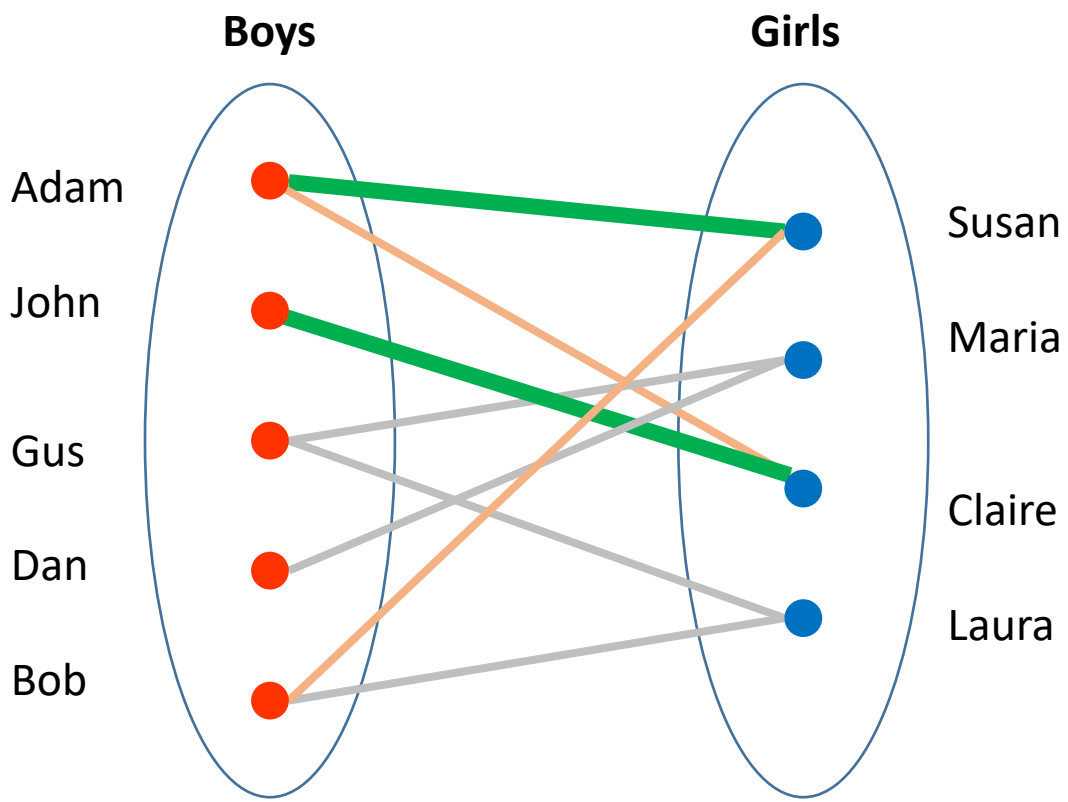

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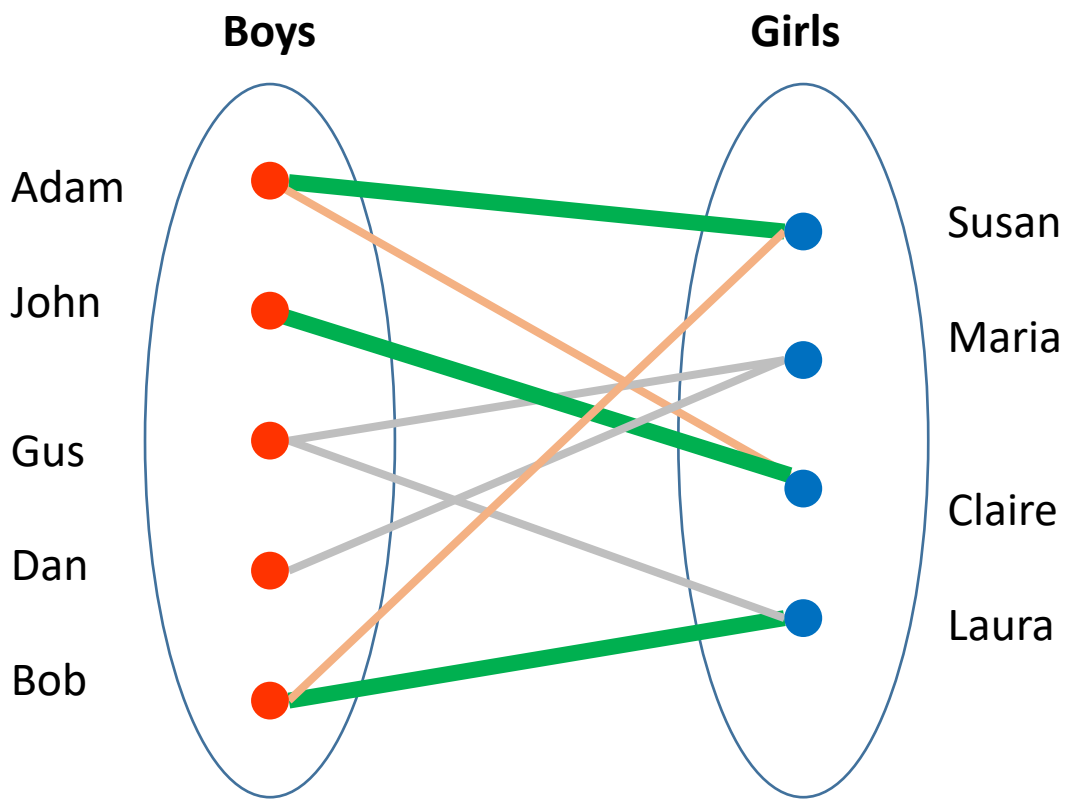

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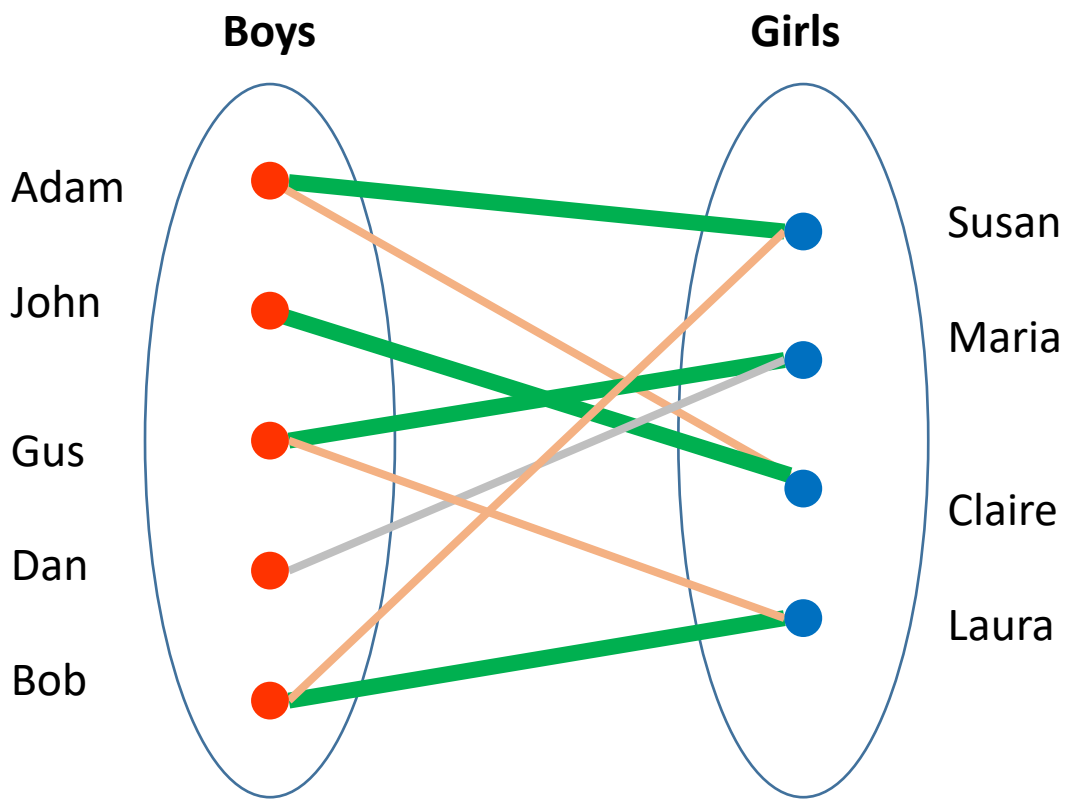
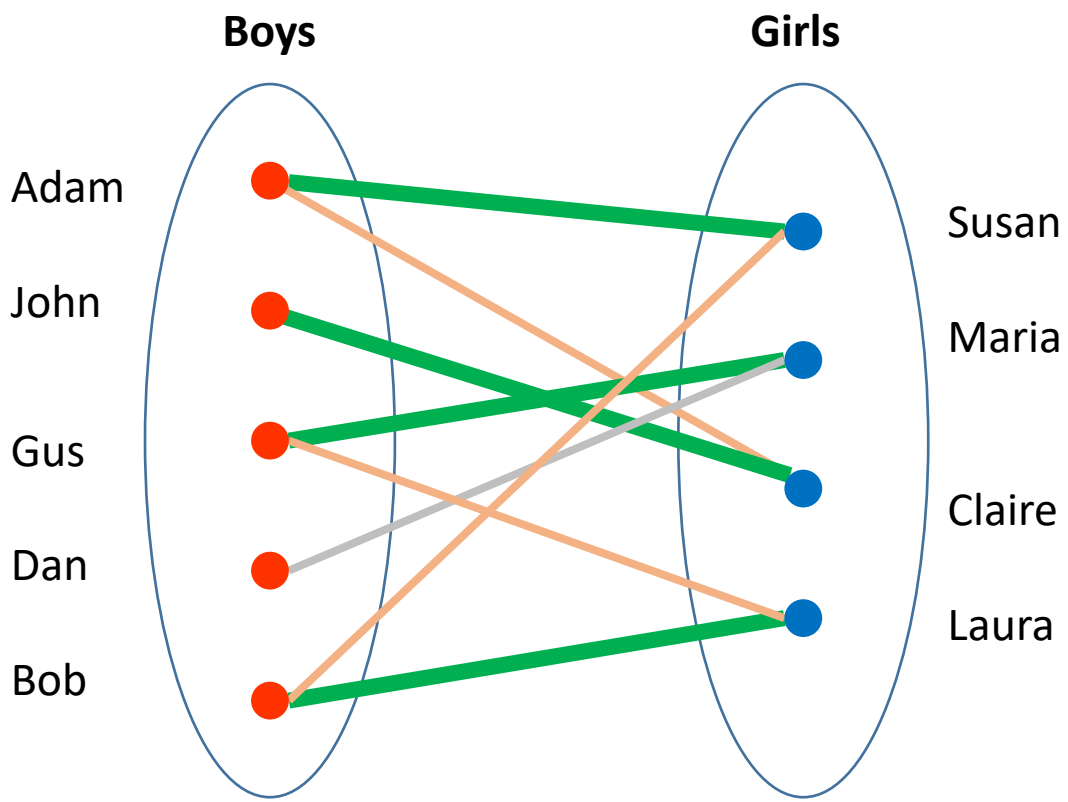


How to Find a Maximum Bipartite Matching?

Part of matching

Not part of matching

- We have found a path whose edges alternate between being part of the matching and not being part of the matching.



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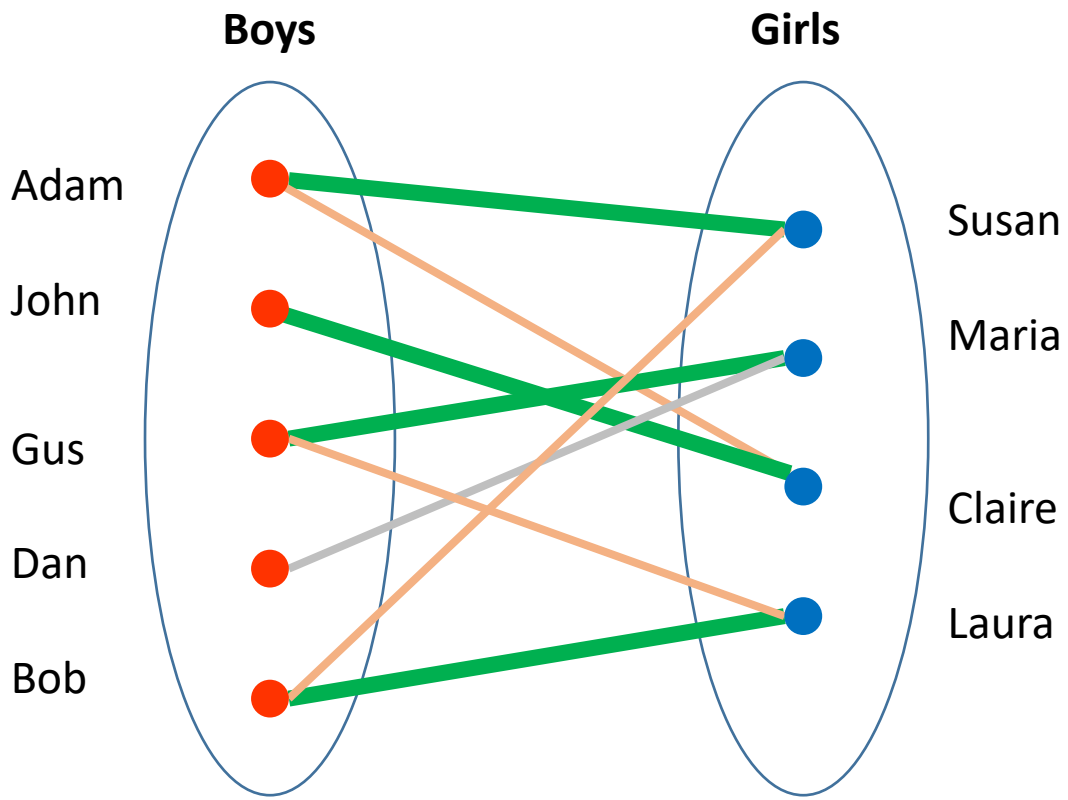
Part of matching

Not part of matching

- We have found a path whose edges alternate between being part of the matching and not being part of the matching.
- Let us now try and generalize this idea.



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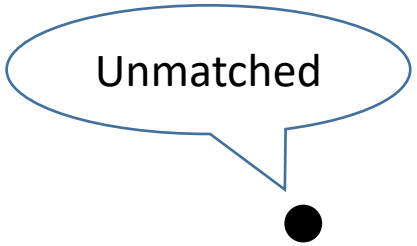

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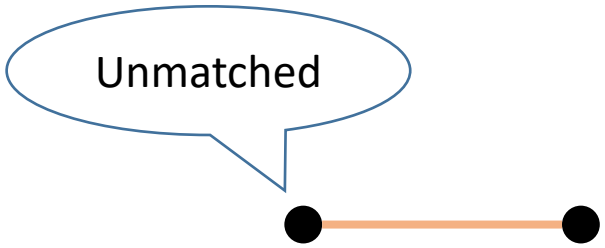

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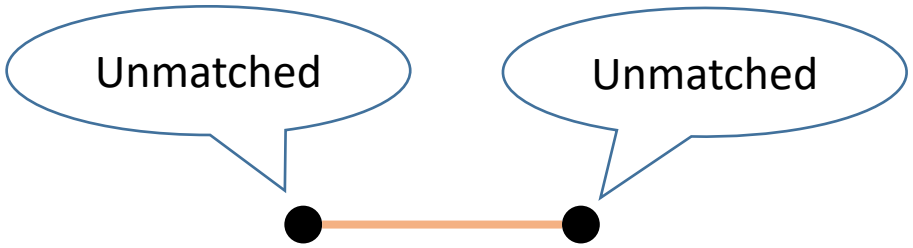

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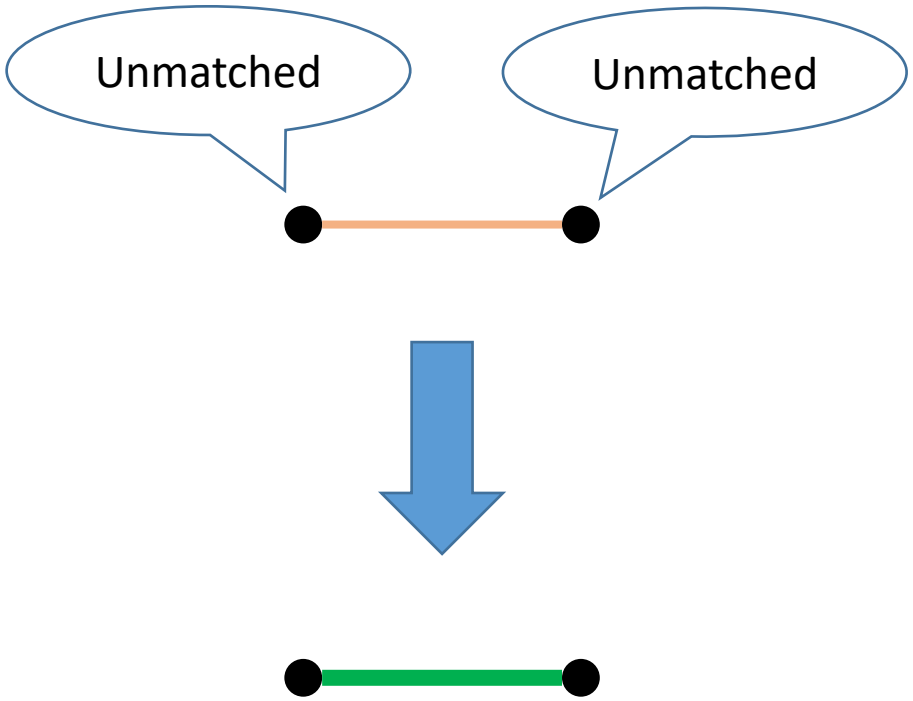

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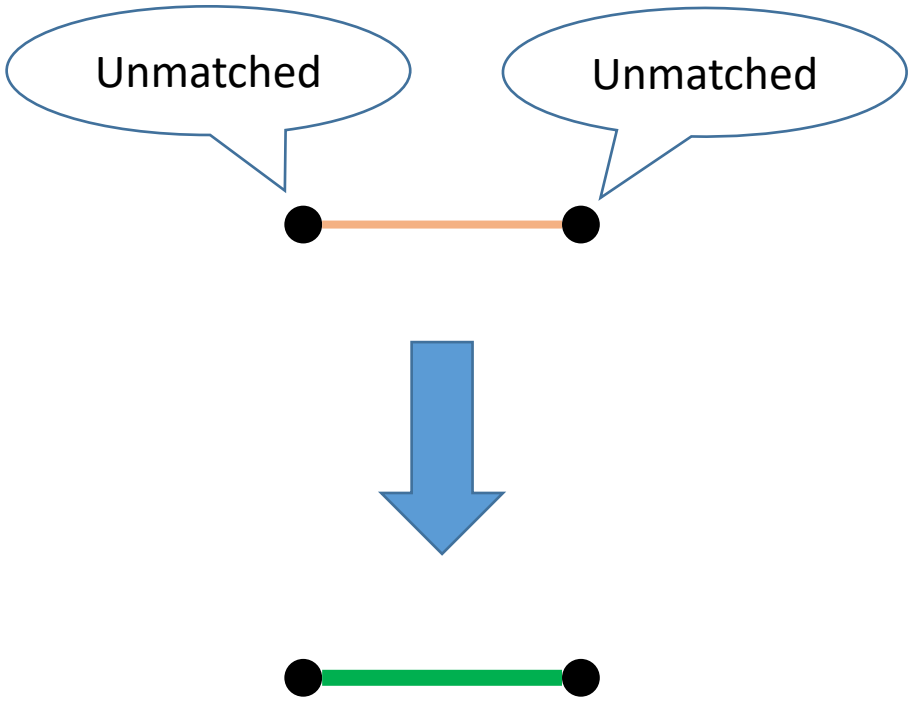
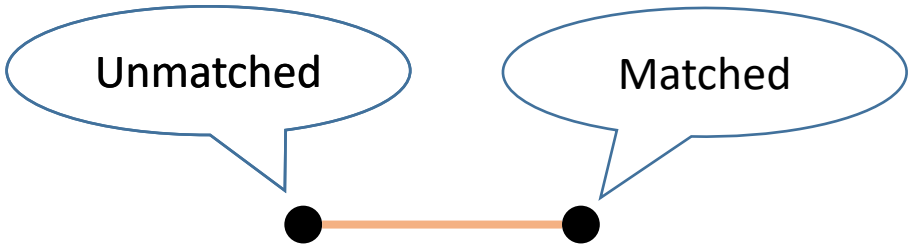


Size of matching increases by one.

Part of matching

Not part of matching

How to Find a Maximum Bipartite Matching?



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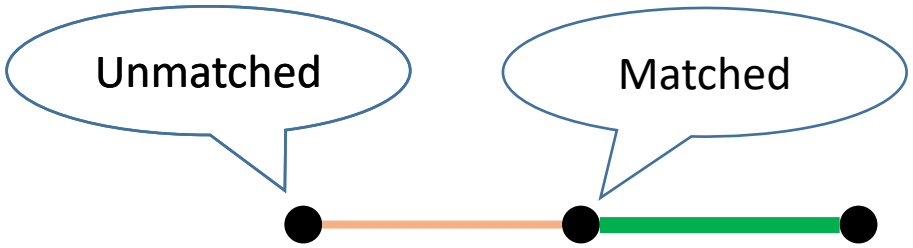

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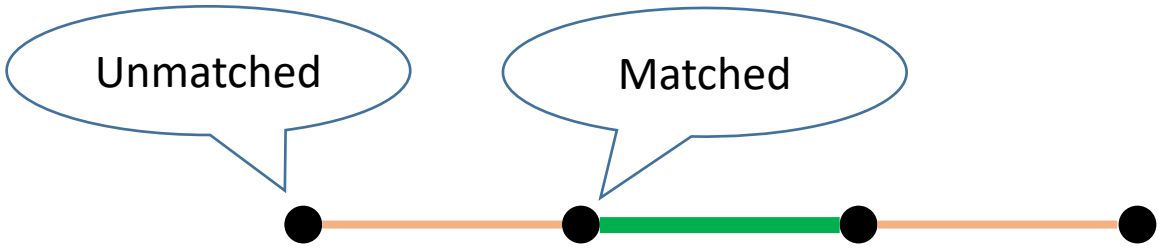

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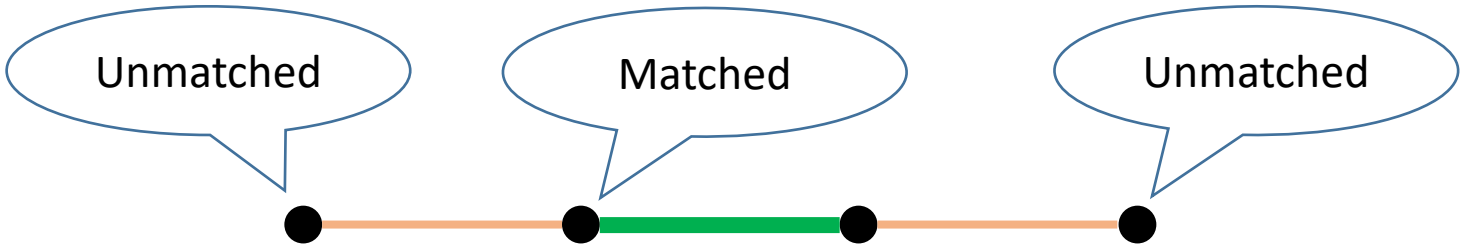

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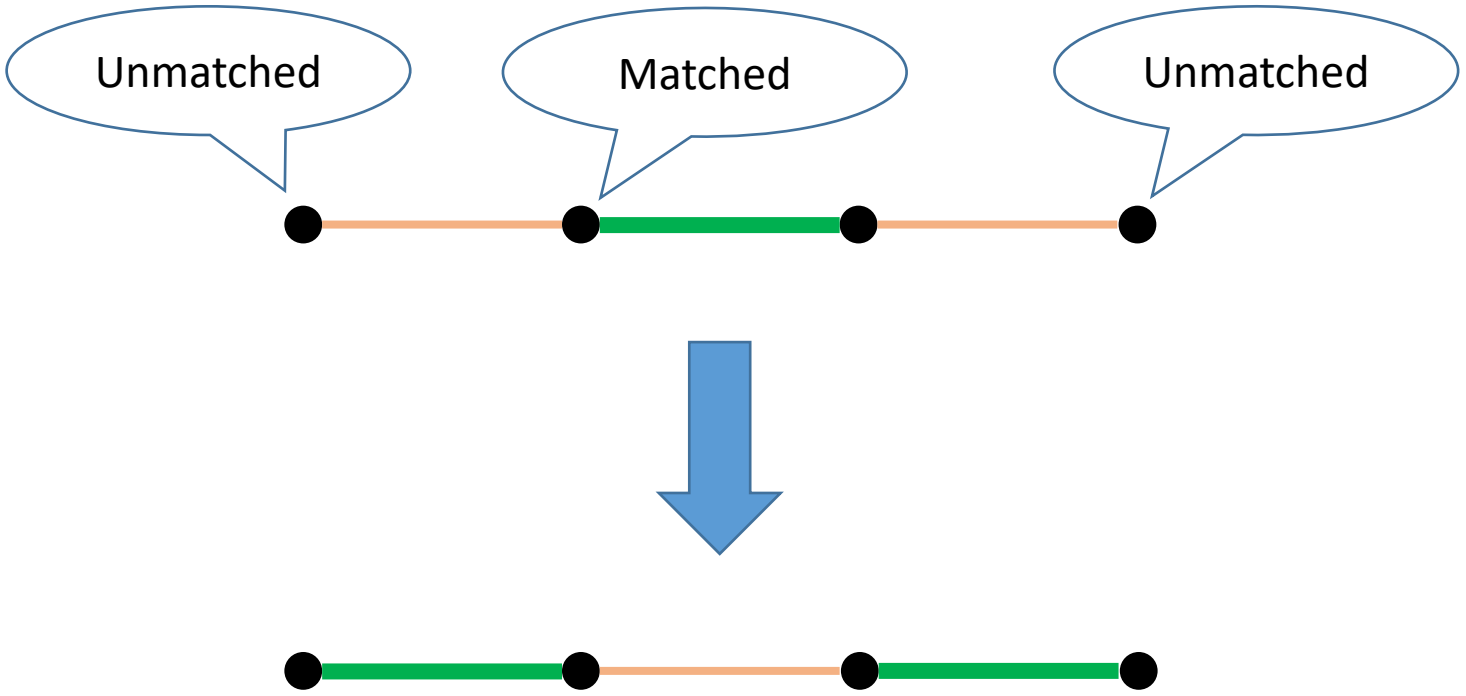

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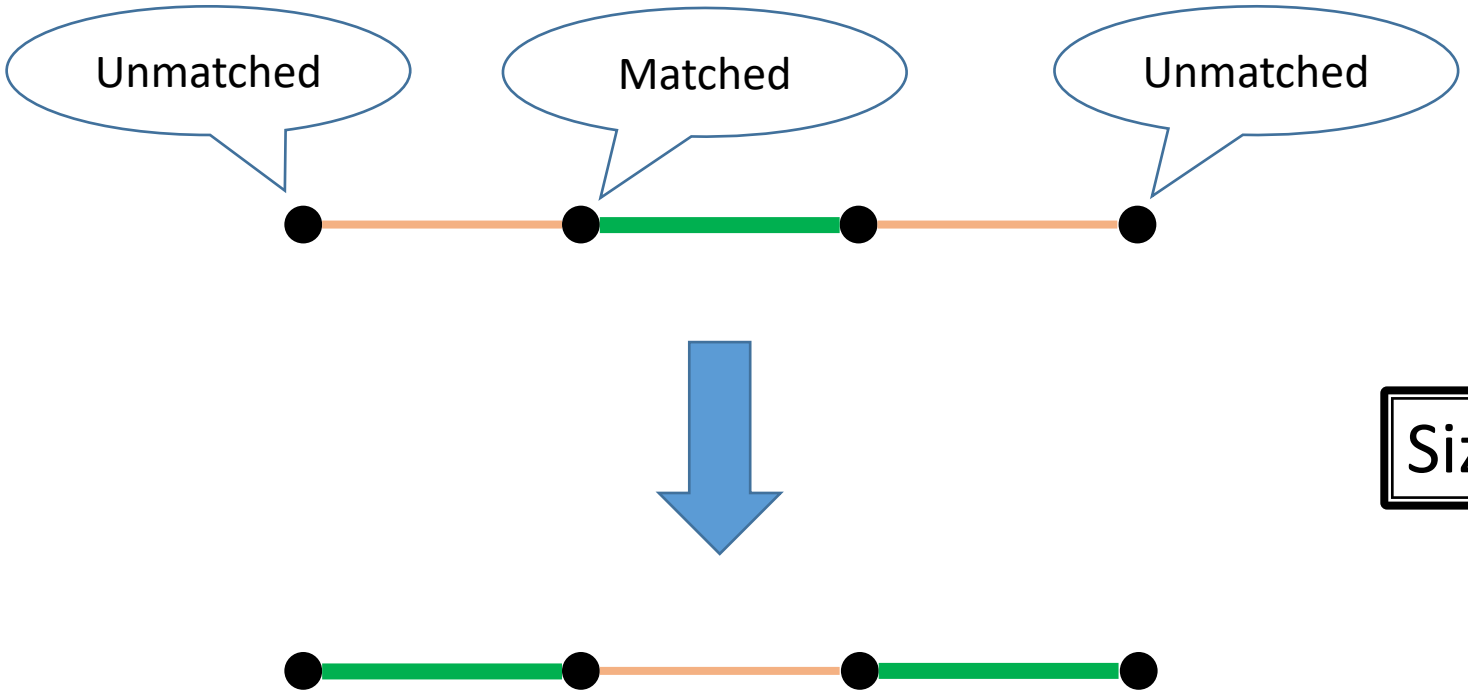

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Part of matching

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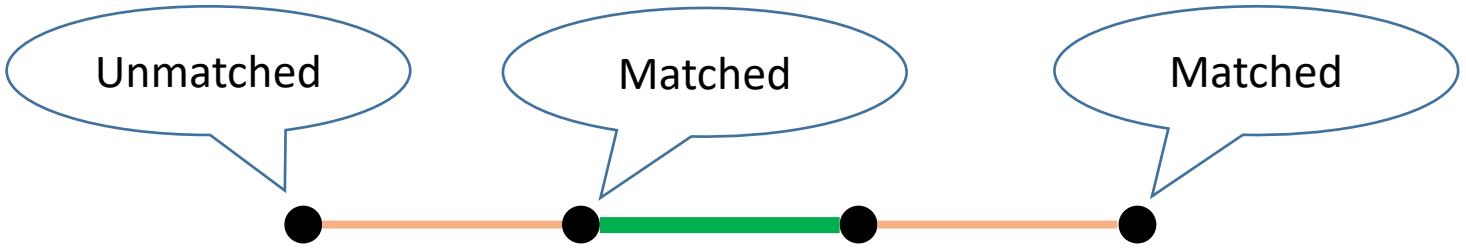

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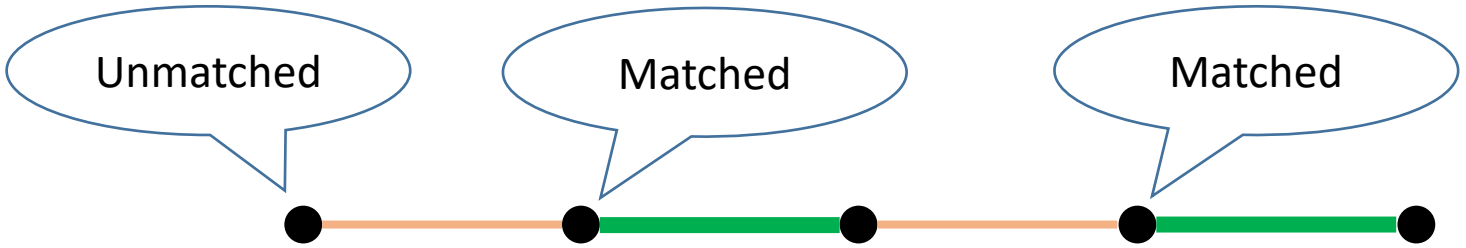

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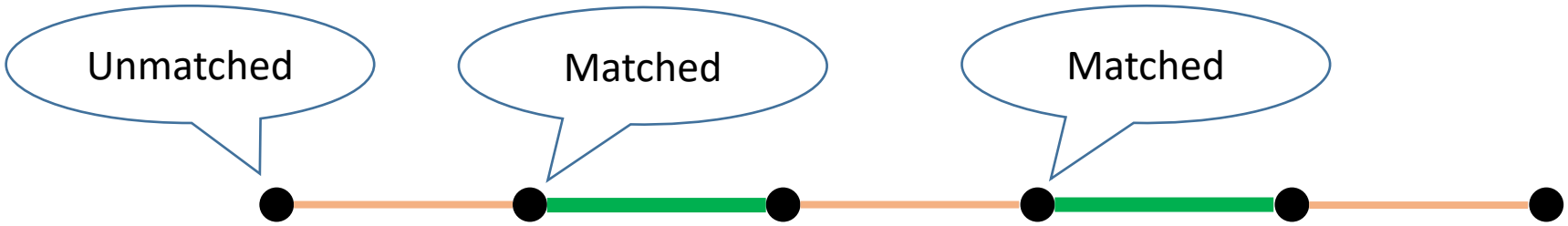

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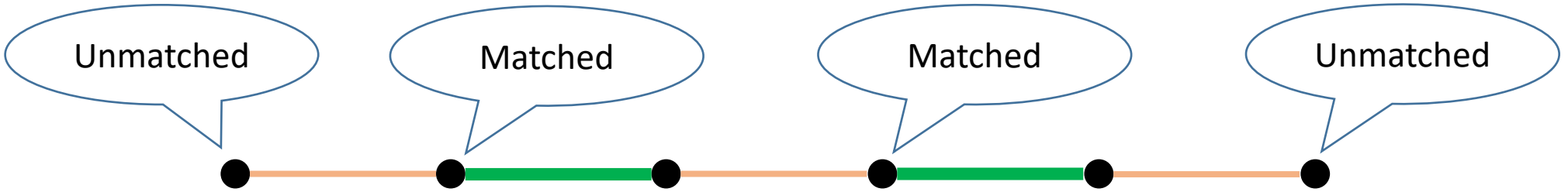

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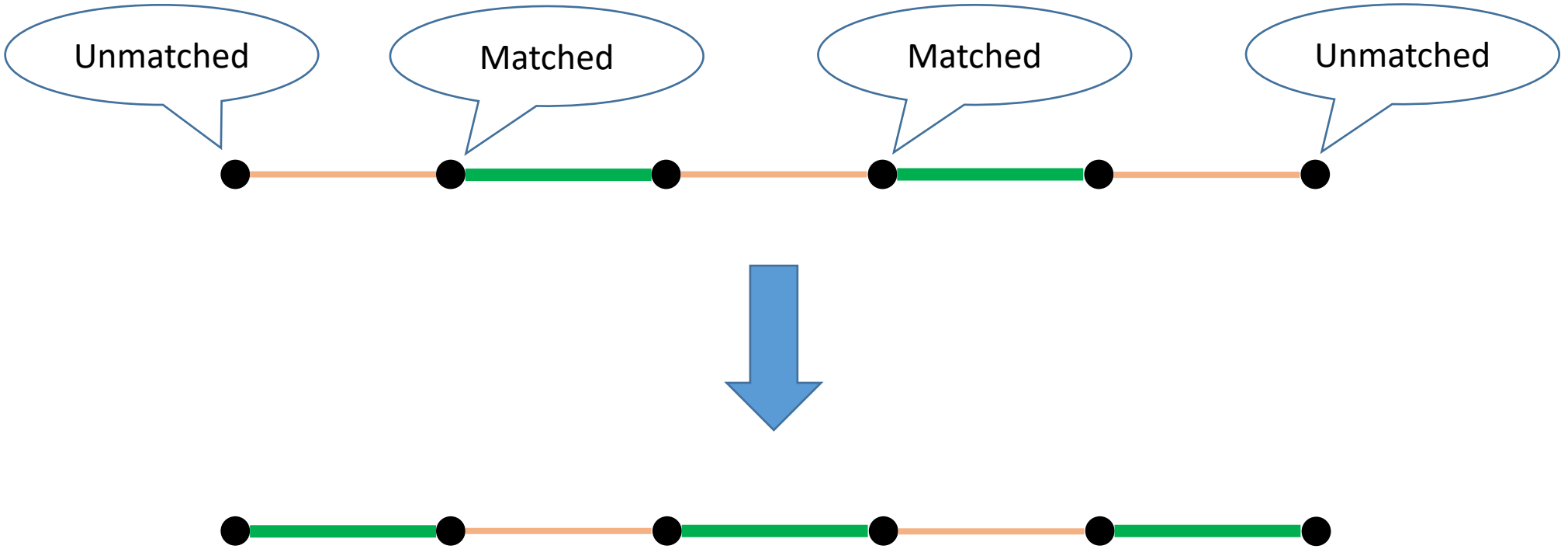

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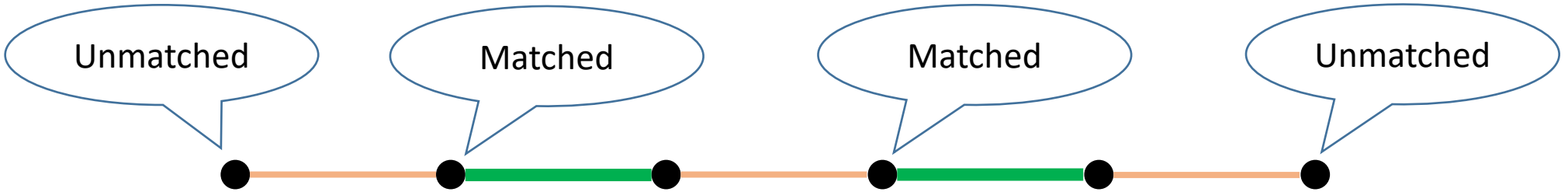

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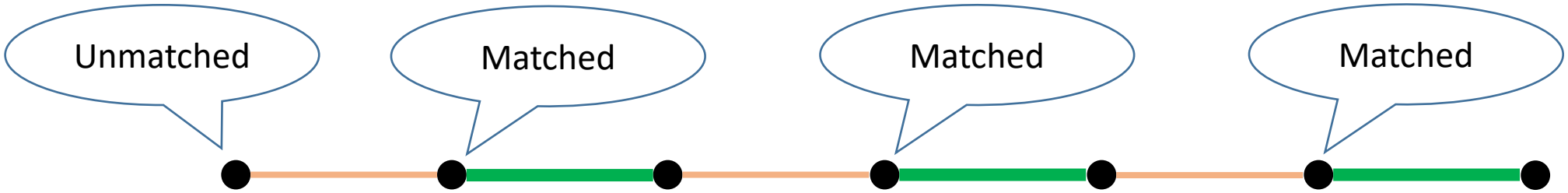

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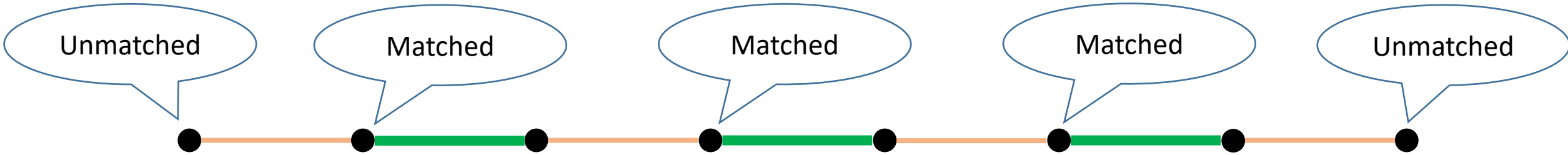

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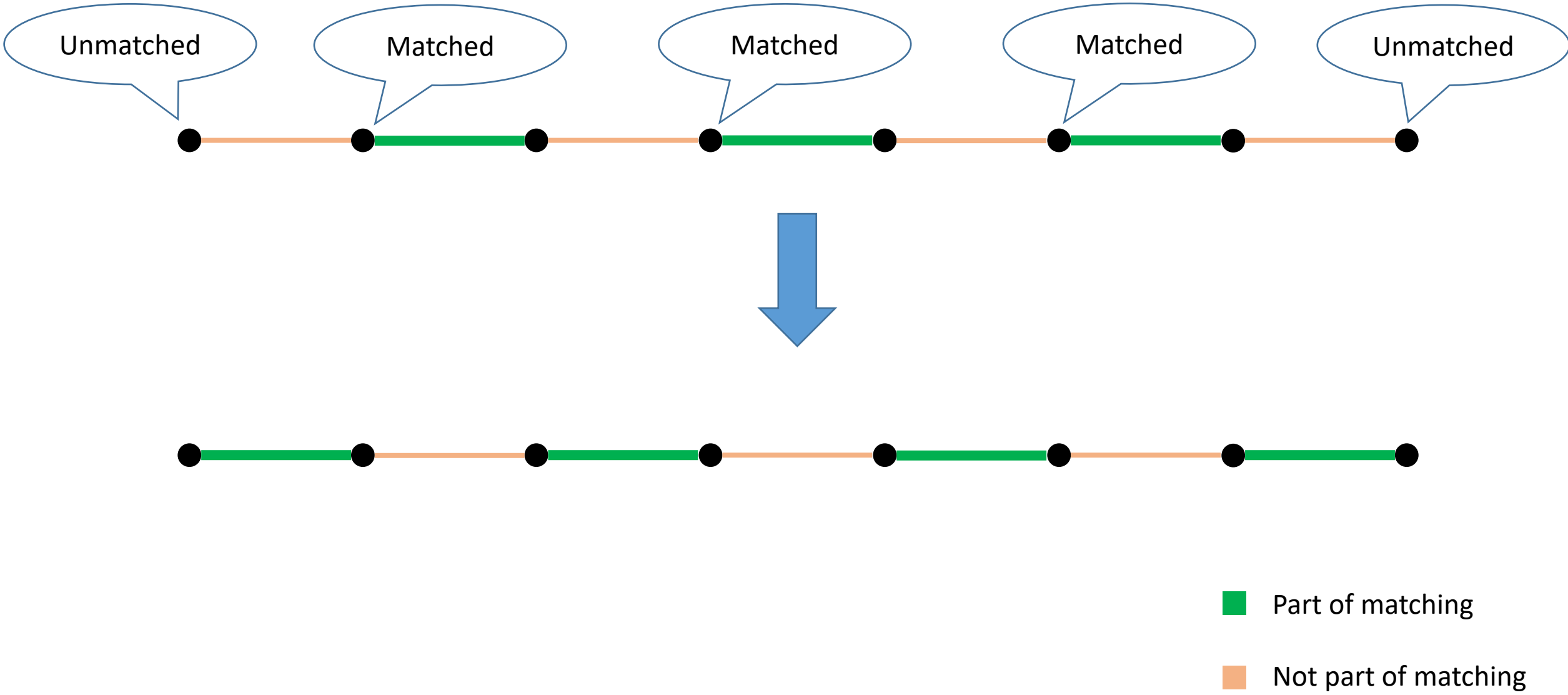

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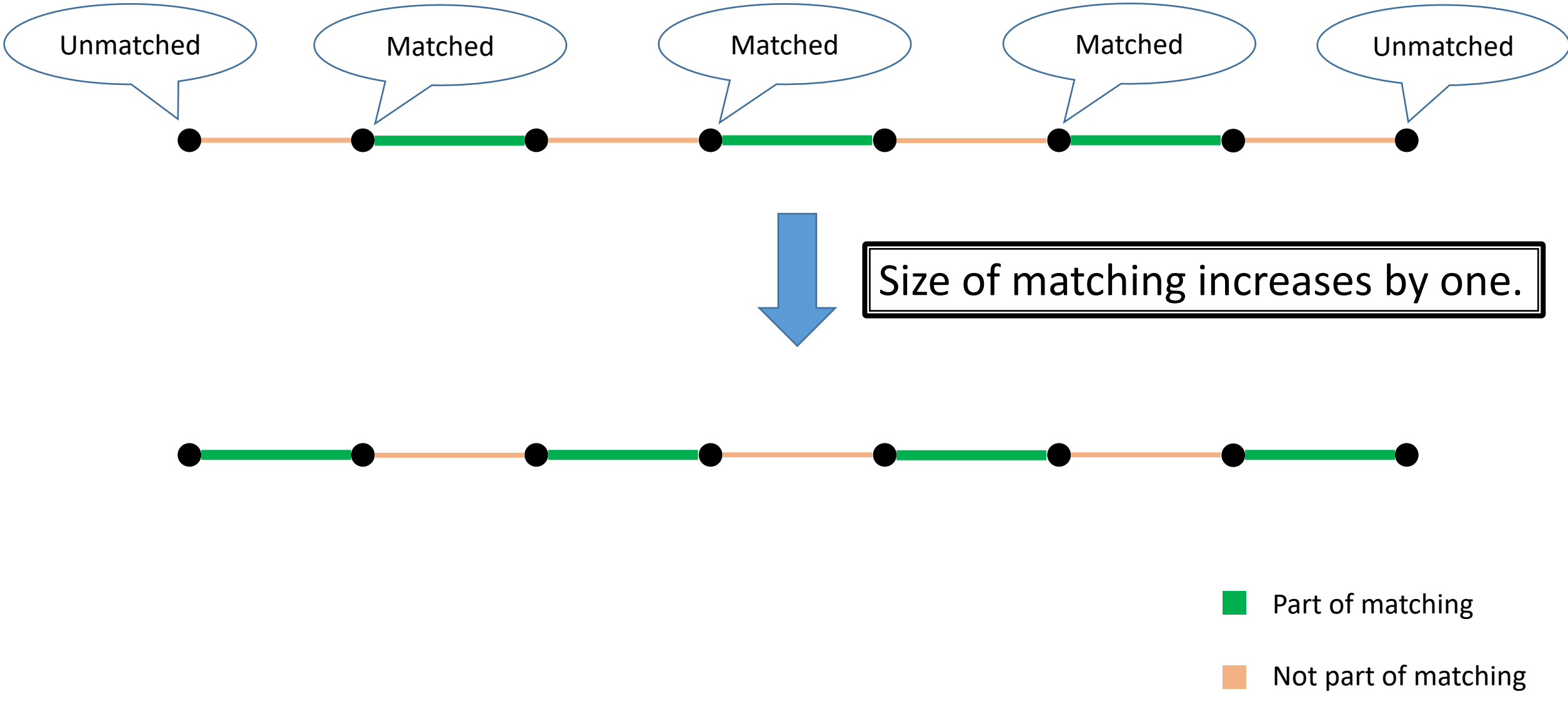

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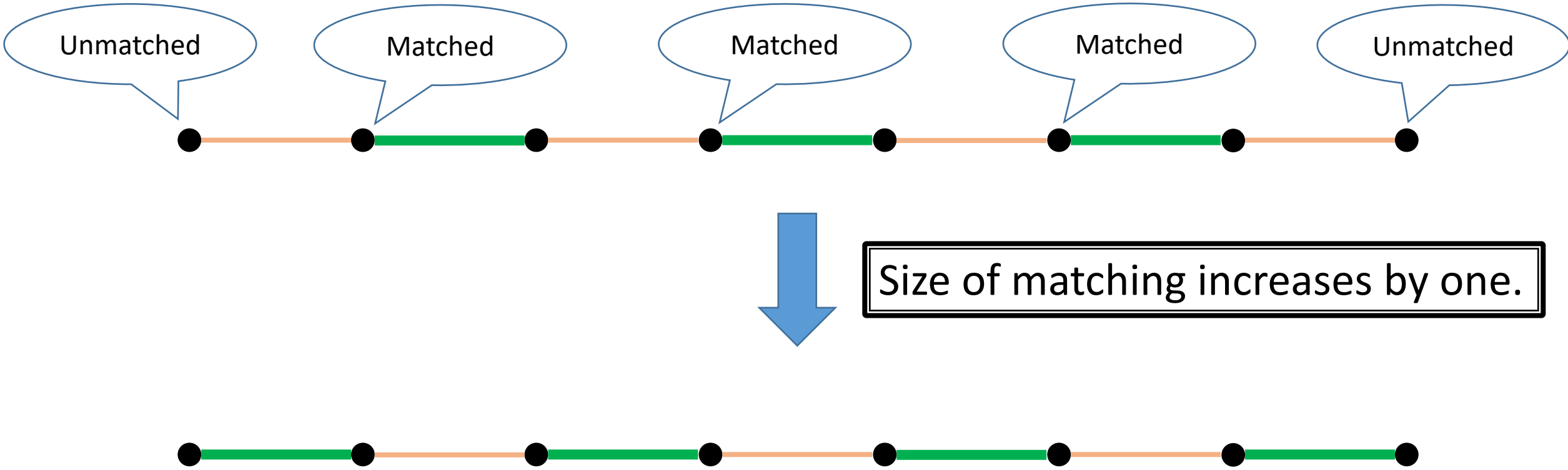

Size of matching increases by one.

Fact: If the first and last vertices on a path are unmatched, the size of the matching can be increased.

Part of matching

Not part of matching

How to Find a Maximum Bipartite Matching?

- Such paths (alternating edges, first and last vertices are unmatched) are called augmenting paths.
- If there is an augmenting path, size of the matching can be increased.
- Can we reach a maximum matching using this method?
- Yes (using Berge's lemma)!

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Berge's lemma: A matching is a maximum matching if and only if it has no augmenting paths.

Proof of Berge's Lemma

- We already saw that if a graph has an augmenting path, the matching is not a maximum matching.
- We will show if a matching is not maximum, it has an augmenting path.
- If a matching M is not maximum, let OPT be a maximum matching.
- Look at the edges of $M \oplus OPT$.

Fact: If the first and last vertices on a path are unmatched, the size of the matching can be increased.

Berge's lemma: A matching is a maximum matching if and only if it has no augmenting paths.

Proof of Berge's Lemma

- In a matching, every vertex has degree at most 1.
- Every vertex of $M \oplus OPT$ has degree at most 2.
- Thus, the graph $M \oplus OPT$ has isolated vertices, paths, and cycles.
- Edges on the paths and the cycles alternate between edges of M and edges of OPT.

Fact: If the first and last vertices on a path are unmatched, the size of the matching can be increased.

Berge's lemma: A matching is a maximum matching if and only if it has no augmenting paths.

Proof of Berge's Lemma

- Edges on the paths and the cycles alternate between edges of M and edges of OPT.
- Thus, the cycles of $M \oplus OPT$ must all be of even length.
- So, all the cycles have an equal number of edges from M and OPT.
- However, we know that $|OPT| > |M|$.

Fact: If the first and last vertices on a path are unmatched, the size of the matching can be increased.

Berge's lemma: A matching is a maximum matching if and only if it has no augmenting paths.

Proof of Berge's Lemma

- However, we know that $|OPT| > |M|$.
- The only way OPT has more edges than M is if there is a path with an odd number of edges, starting and ending with edges of OPT.
- And, this is precisely the definition of an augmenting path for M.
- Hence, if a matching is not maximum, it has an augmenting path.

Fact: If the first and last vertices on a path are unmatched, the size of the matching can be increased.

Berge's lemma: A matching is a maximum matching if and only if it has no augmenting paths.

How to Find a Maximum Bipartite Matching?

- Berge's lemma tells us that our approach of only looking for augmenting paths is correct. But, a graph can have exponentially many paths.
- So, how do we find augmenting paths efficiently?
- The Hopcroft-Karp algorithm for maximum matching in bipartite graphs!
- What does a BFS traversal tree in a bipartite graph look like?

Fact: If the first and last vertices on a path are unmatched, the size of the matching can be increased.

Berge's lemma: A matching is a maximum matching if and only if it has no augmenting paths.

Hopcraft-Karp Algorithm

Input: Bipartite graph $G(U \cup V, E)$

Output: Matching $M \subseteq E$

$M \leftarrow \emptyset$

repeat

$\mathcal{P} \leftarrow \{P_1, P_2, \ldots, P_k\}$ *maximal set of vertex-disjoint shortest augmenting paths*

$M \leftarrow M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$

until $\mathcal{P} = \emptyset$

Thank You!