Popular Matchings - Part I

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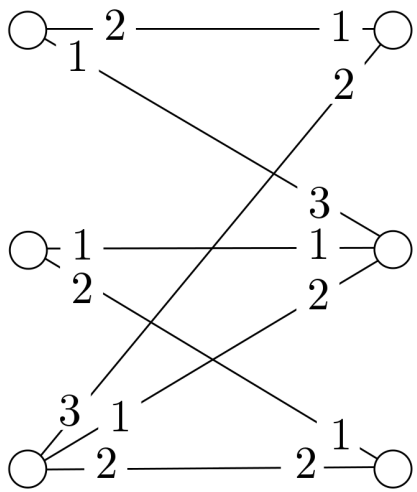
The input

A bipartite graph G where every vertex v has a strict ranking $\succ_v$ of its neighbors.



A well-studied model used in many two-sided markets:

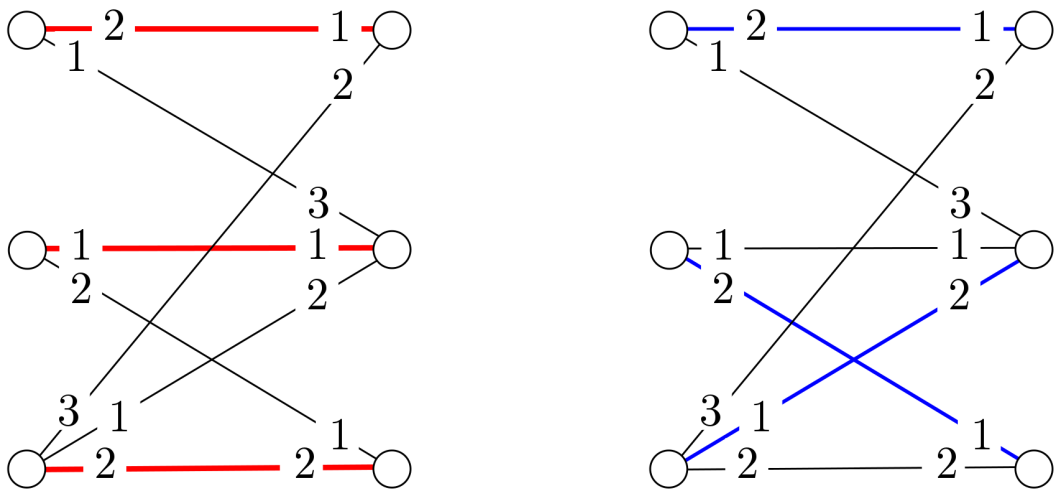
- students to schools;
- medical residents to hospitals;
- agents to jobs.

Stability

A matching M is stable if there is no edge ab such that:

$$b \succ_a M(a) \qquad \text{and} \qquad a \succ_b M(b)$$

(i.e., a and b prefer each other to their respective assignments in M)



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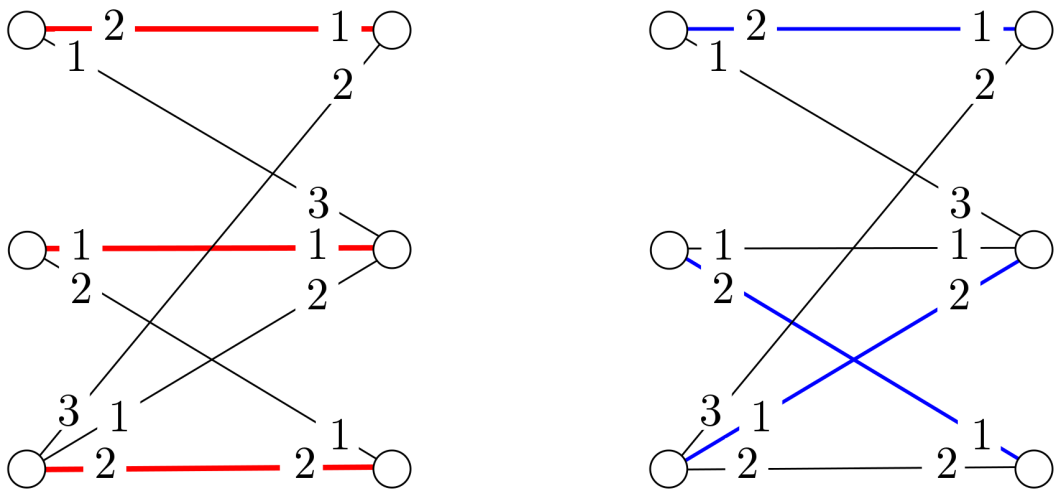

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- The red matching is stable but the blue one is not.

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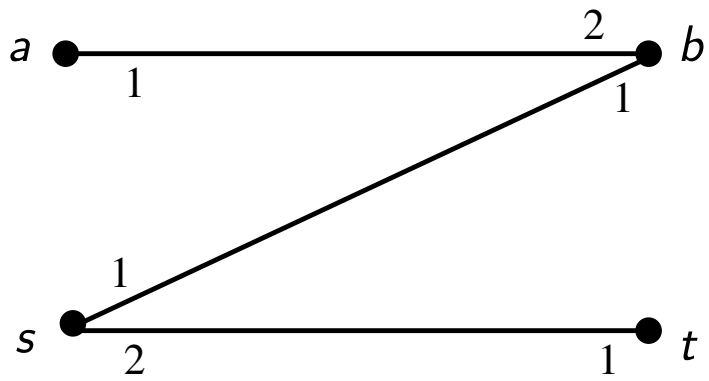

Let us run Gale-Shapley algorithm on this instance.

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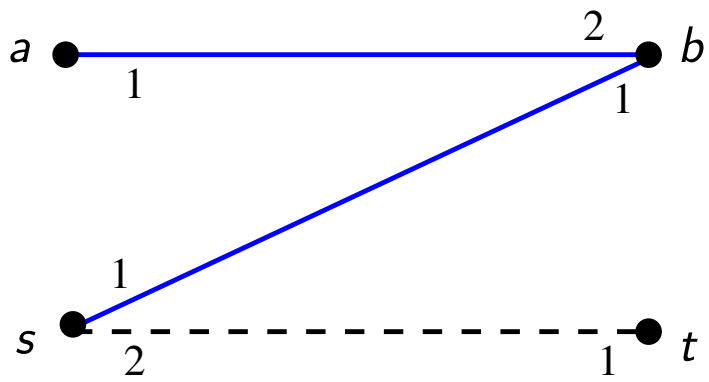

Initially both a and s propose to their top neighbour b.

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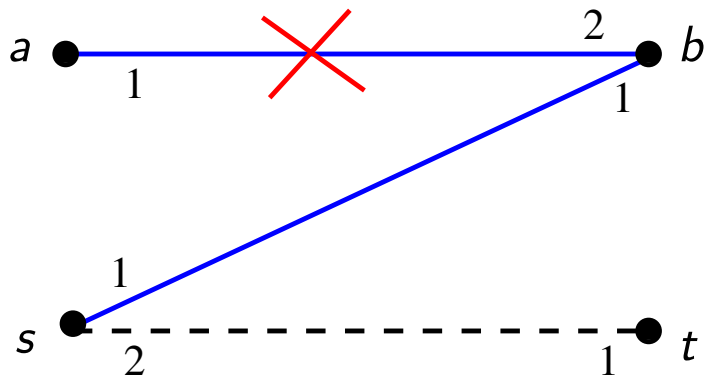

b rejects a's proposal and (tentatively) accepts s's proposal.

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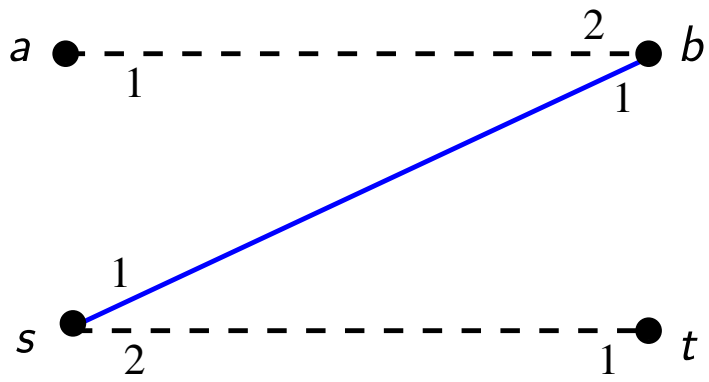

a has no other neighbour to propose to; we get the matching $\{sb\}$.

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Stable matchings are used in several problems in economics, computer science, and operations research.

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To match students to schools in New York:

- How Game Theory Helped Improve New York City's High School Application Process, New York Times, December 5, 2014.

To match students to colleges in France:

- Stable Matching in Practice, Claire Mathieu. ESA 2018, Keynote talk.

To match students to engineering colleges in India:

- Centralized admissions for engineering colleges in India, S. Baswana, P. P. Chakrabarti, S. Chandran, Y. Kanoria, and U. Patange. INFORMS Journal on Applied Analytics, 2018.

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To match medical residents to hospitals:

- National Resident Matching Program `http://www.nrmp.org/whythematch.pdf`
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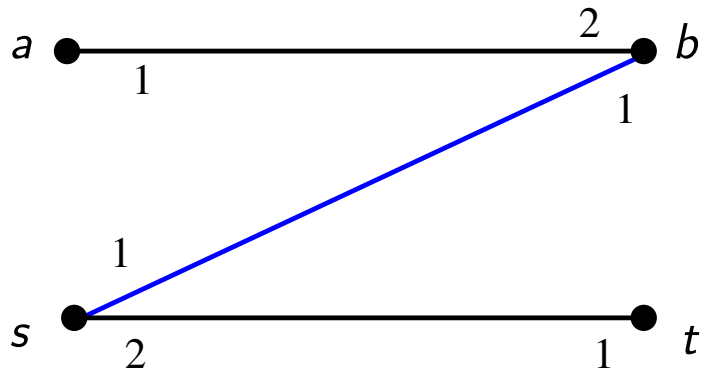
In all these applications, we want a large matching.

All stable matchings match the same subset of vertices (Rural Hospitals Theorem).

Size versus Stability

The size of a stable matching could be as low as $|M_{\max}|/2$.



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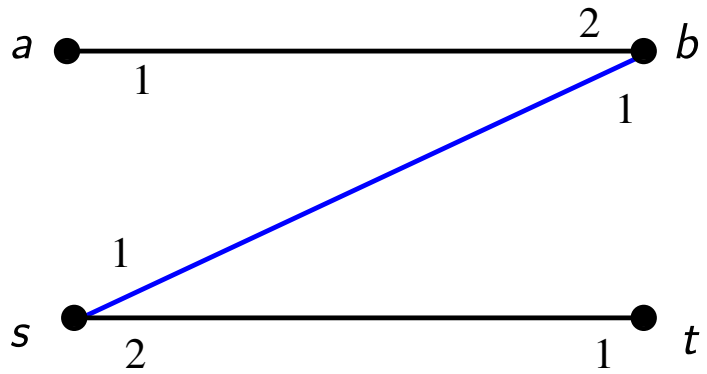

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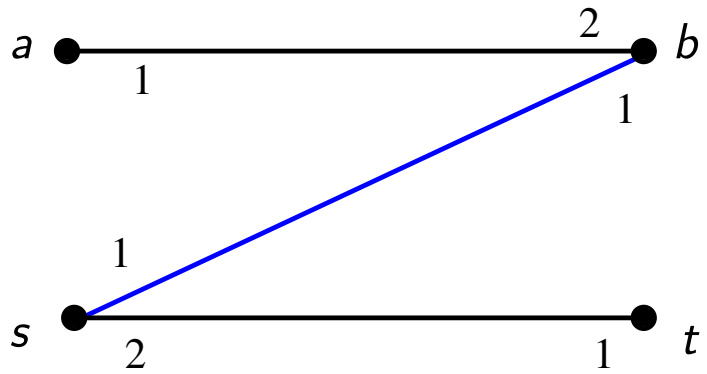

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Drawbacks of stability:

- Size can be half the size of a max-size matching;
- Models a situation where every edge has a "veto power".

Beyond stability

Can we relax stability so as to cope with these issues? We want a set that:

- contains stability as a special case;
- shifts the focus from "veto power" to "collective decision";
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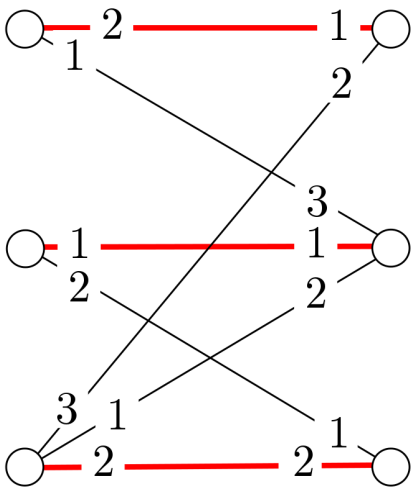
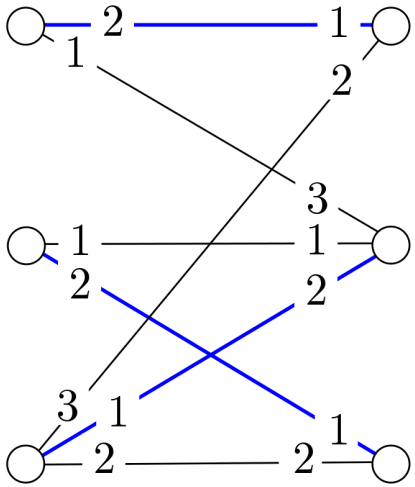
- Any pair of matchings M and N can be compared via an M-vs-N election.
  - Every vertex votes for the matching in $\{M, N\}$ that it prefers.

Elections between pairs of matchings

Consider the election between the red and blue matchings.





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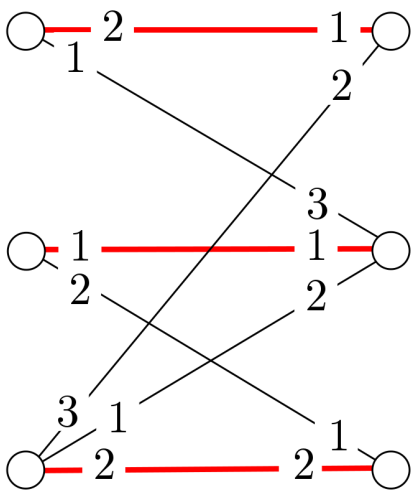

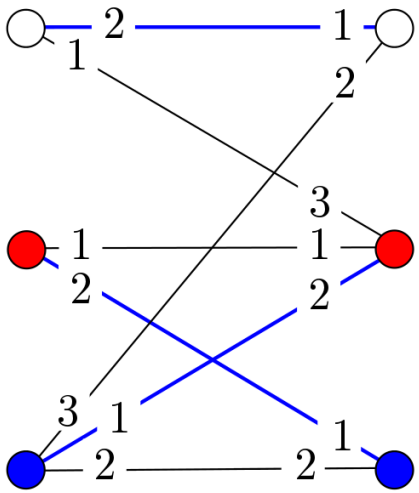

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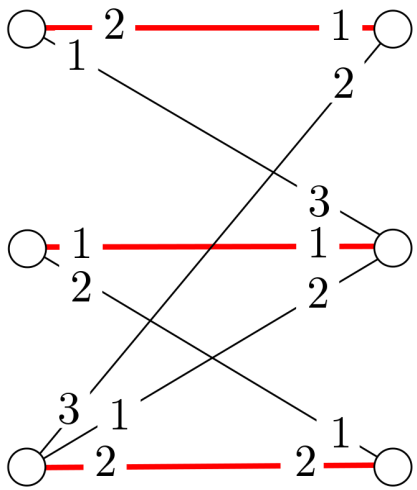

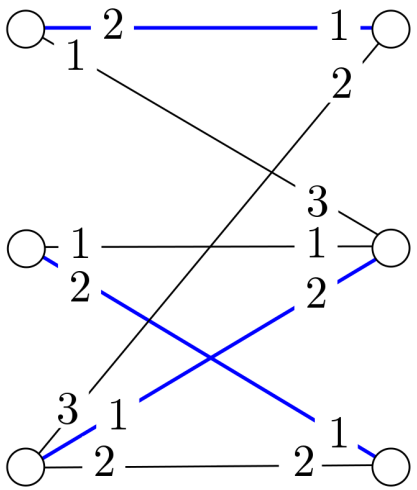

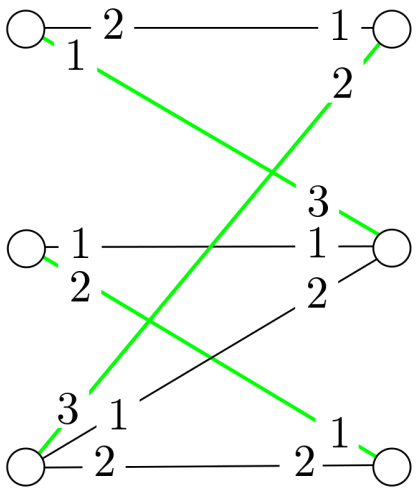

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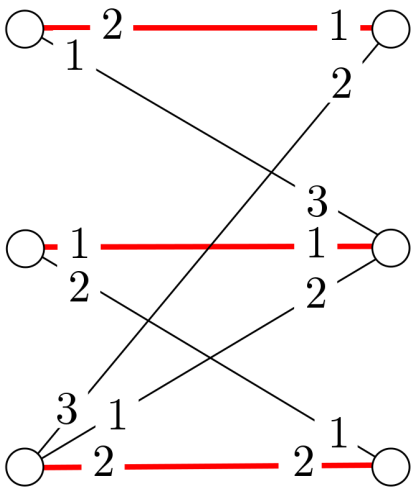

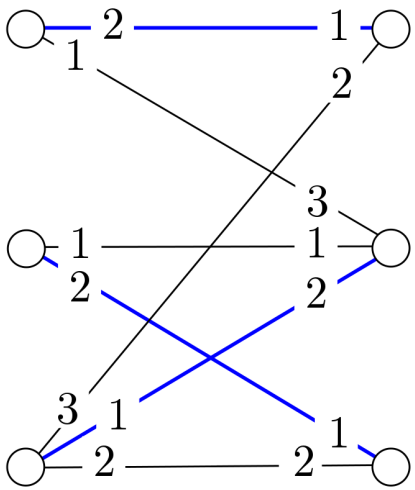

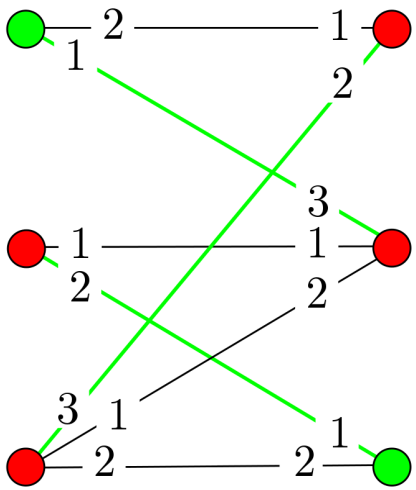

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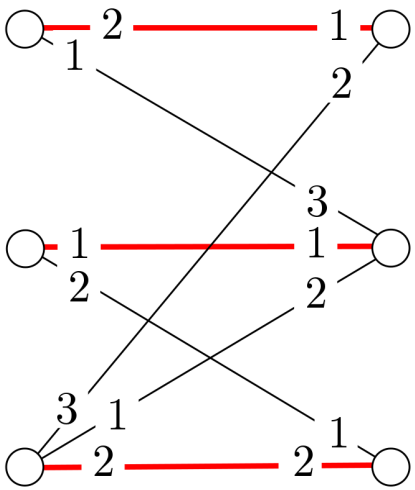

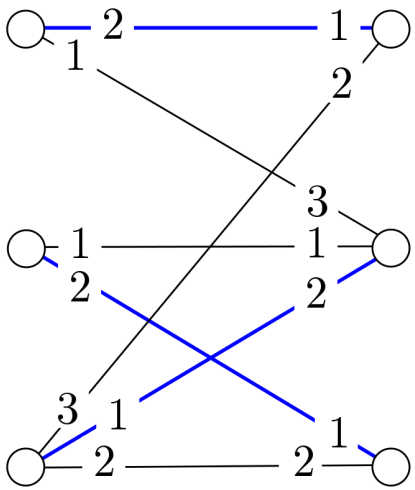

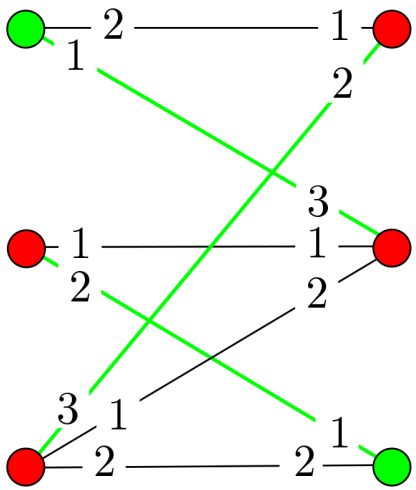

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A popular matching is one that does not lose any election.

Condorcet winner

Condorcet winner: A candidate who defeats every other candidate in their head-to-head election.

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Comparing a stable matching S with any matching N:

- u prefers N to S $\implies$ $N(u)$ has to prefer S to N;

  (otherwise the edge between u and $N(u)$ blocks S)
- so # of votes for $N \leq$ # of votes for S.

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Is there an efficient algorithm to find a max-size popular matching?

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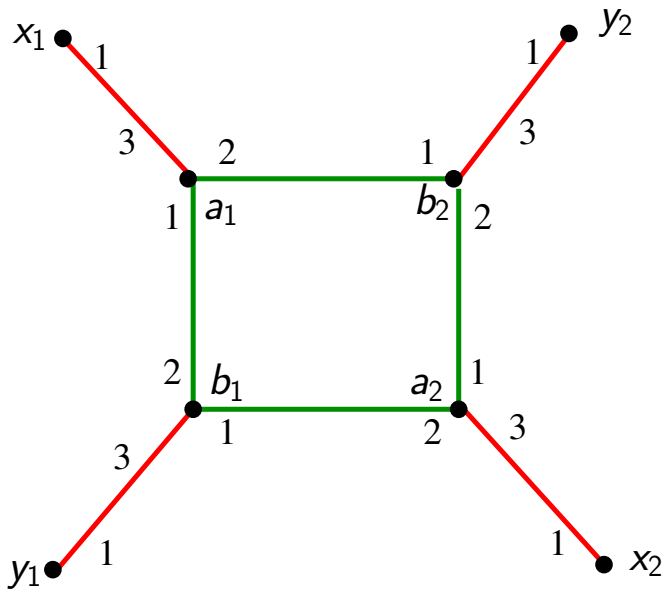
The invariant should be that M is a popular matching in G.

An interesting example

A stable matching has size 2 here while a max-size popular matching has size 4.



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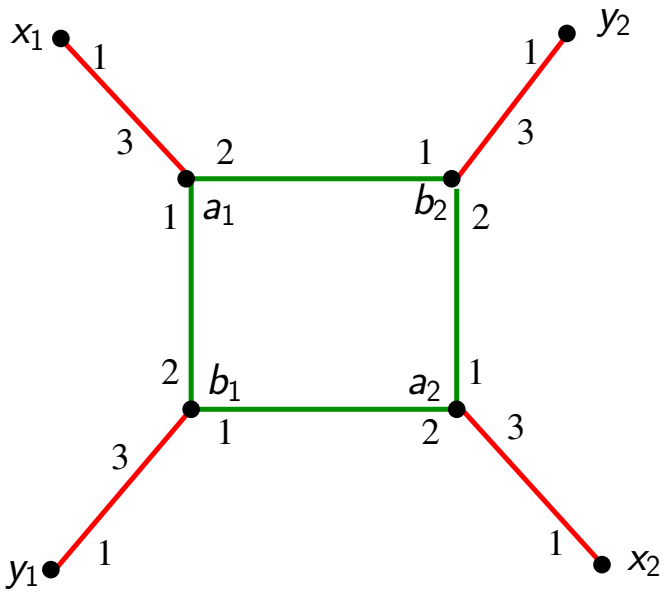

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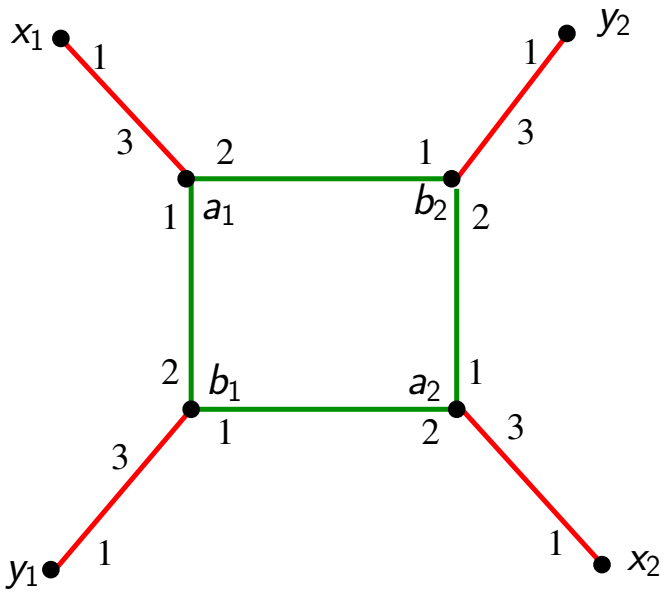

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To find a max-size popular matching, can we adapt the Gale-Shapley algorithm?

To find a max-size popular matching

Stability is easy to check: no edge blocks a stable matching.

Popularity requires comparing our matching with all the matchings in G.

- Does a popular matching admit a simple *certificate* of its popularity?
- How do we show that *no* larger matching is popular?

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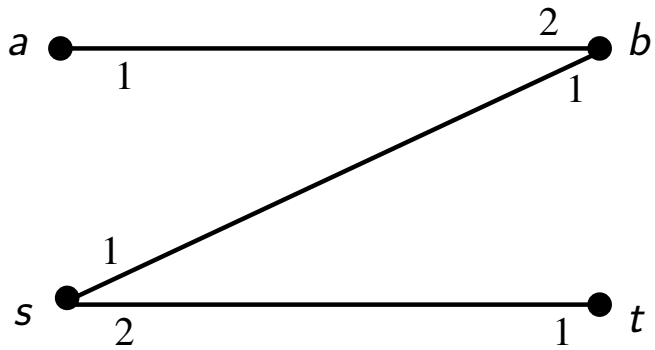
Let G be the graph below:



We want to find the matching $\{ab, st\}$.

- This is a max-size popular matching in this instance.

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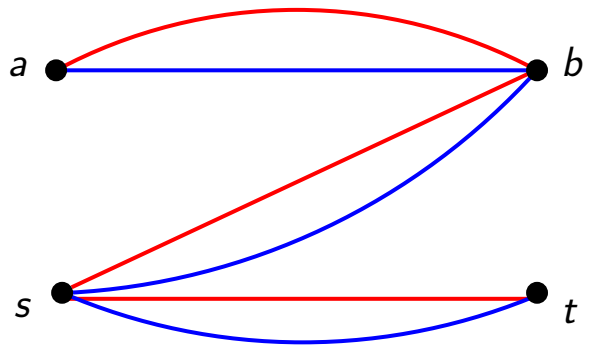

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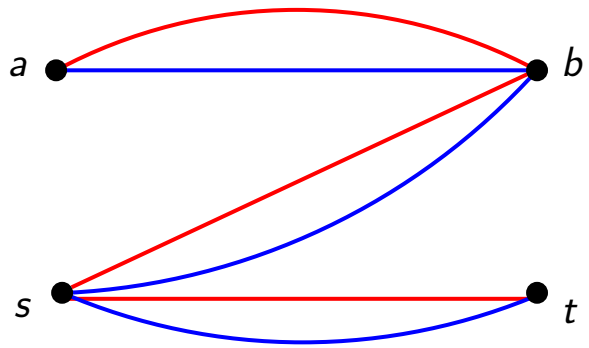

- Every vertex in A prefers any red edge to any blue edge.
- Every vertex in B prefers any blue edge to any red edge.

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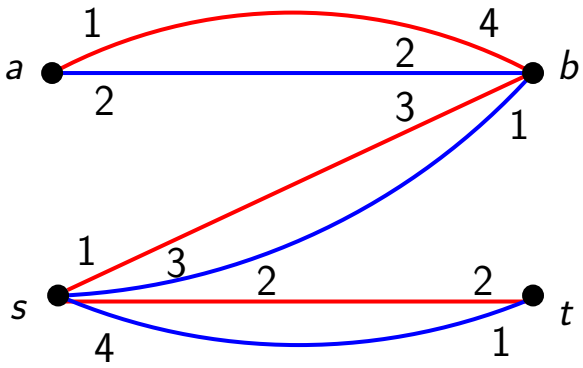

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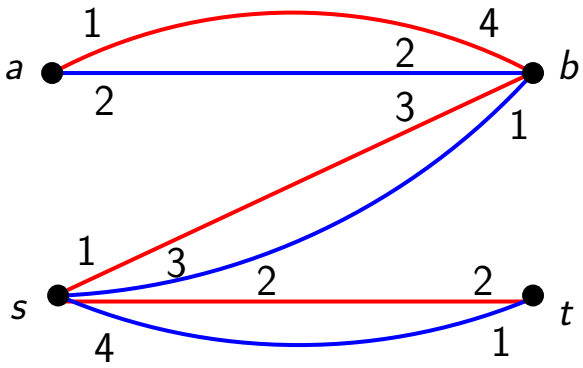

- The preference order of s in G is $sb \succ_s st$. Its preference order in G' is:

$$sb \succ_s st \succ_s sb \succ_s st.$$

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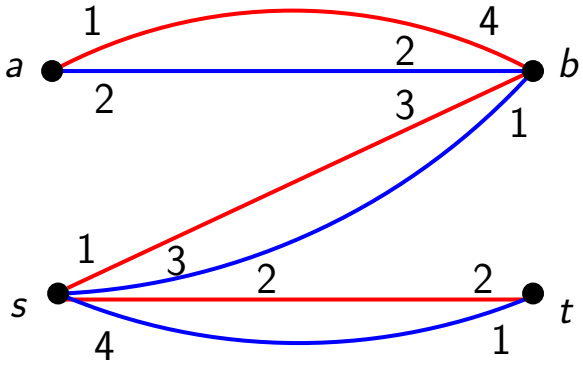

- The preference order of s in G is $sb \succ_s st$. Its preference order in G' is:

$$sb \succ_s st \succ_s sb \succ_s st.$$

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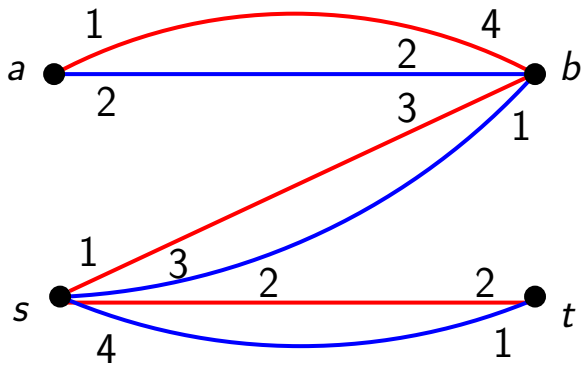

Recall the stable matching $\{sb\}$ in G.

- In the graph G', neither $\{sb\}$ nor $\{sb\}$ is stable.

A new instance G'

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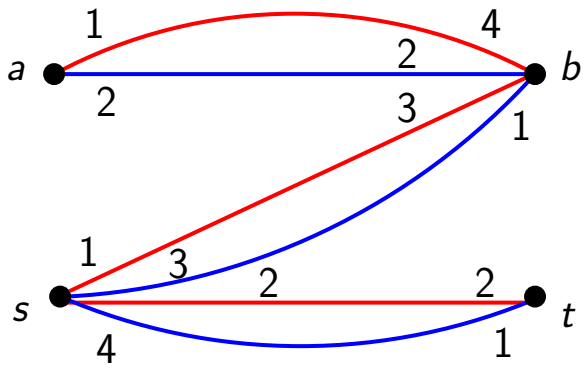
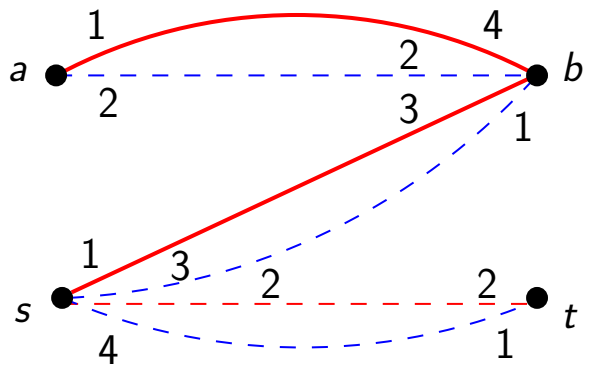


Recall the stable matching $\{sb\}$ in G.

- In the graph G', neither $\{sb\}$ nor $\{sb\}$ is stable.
  - The edge ab blocks the matching $\{sb\}$.
  - The edge st blocks the matching $\{sb\}$.

Computing a stable matching in G'

Let us run Gale-Shapley algorithm in G'.



- Both a and s propose to b along their red edges.

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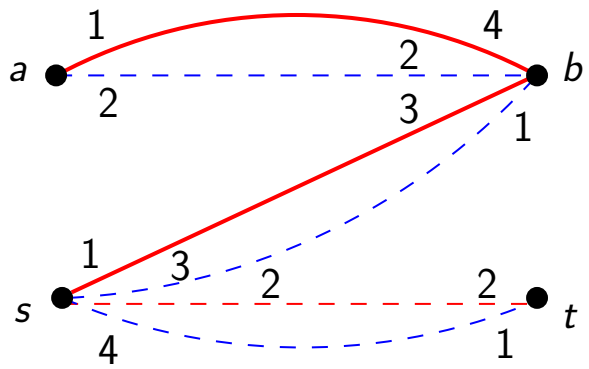

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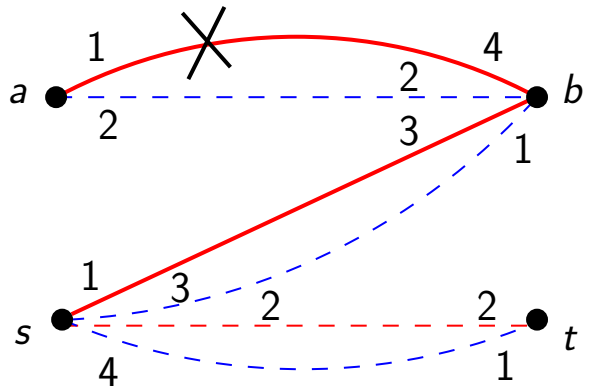

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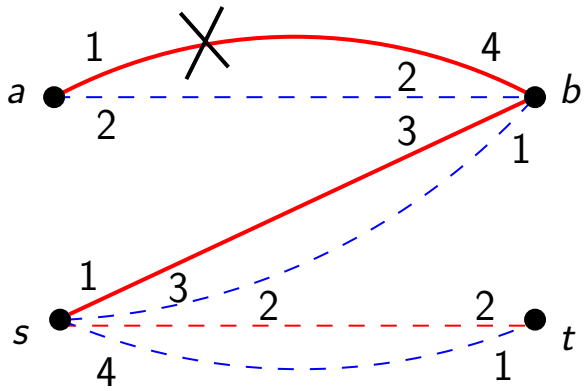

- So b rejects a's proposal and (tentatively) accepts s's proposal.
- Then a proposes along its next favorite edge: this is ab.

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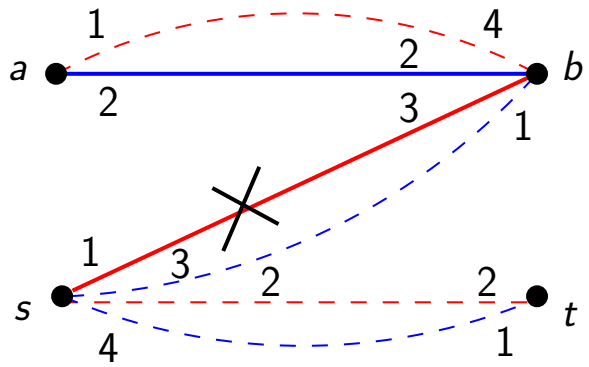

- Observe that now b prefers a's proposal to s's proposal.
- So b rejects s's proposal and (tentatively) accepts a's proposal.

Computing a stable matching in G'

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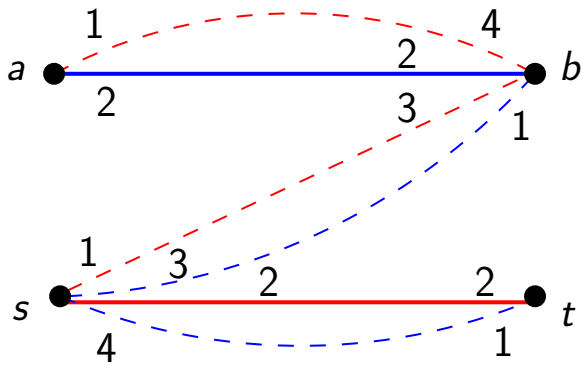

- Then s proposes along its next most favorite edge st.
- t (tentatively) accepts s's proposal. This is the end of the algorithm.

Computing a stable matching in G'

All tentative proposals become definite.

So we get the stable matching $\{ab, st\}$ in G'.

- Ignoring colours, this is the desired matching $M = \{ab, st\}$ in G.

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Our algorithm in $G = (A \cup B, E)$

- Construct the red/blue graph $G' = (A \cup B, E')$.
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CLAIM. M is a max-size popular matching in G.

Analyzing our algorithm

Every popular matching admits a simple certificate of its popularity.

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$$\text{wt}_M(ab) \;=\; \text{vote}_a(b, M(a)) \;+\; \text{vote}_b(a, M(b)).$$

$$\text{Here } \text{vote}_v(u, u') \;=\; \begin{cases} 1 & \text{if } v \text{ prefers } u \text{ to } u' \\ -1 & \text{if } v \text{ prefers } u' \text{ to } u \\ 0 & \text{otherwise.} \end{cases}$$

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So $\mathrm{wt}_M(e) \in \{0, \pm 2\}$ for any edge e.

- Observation. For any edge e, $\mathrm{wt}_M(e) = 2 \iff e$ is a blocking edge to M.

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Let us augment G with self-loops:

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  $\iff$ any perfect matching in G with edge weights given by wt_M has weight at most 0.

LP for max-weight perfect matching

$$\max \sum_{e} \mathrm{wt}_M(e) \cdot x_e$$

$$\begin{aligned} \sum_{e \in \delta(u) \cup \{uu\}} x_e &= 1 \quad \forall u \in A \cup B \\ x_e &\geq 0 \quad \forall e \in E \cup \{\text{self-loops}\}. \end{aligned}$$

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Dual certificate

Every stable matching S has a simple dual certificate: $\vec{\alpha} = \vec{0}$.

- This is because $\text{wt}_S(e) \leq 0$ for all edges e.

Does M computed by our algorithm have an easy-to-describe dual certificate?

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- a is matched along a red edge in M': set $\alpha_a = 1$.
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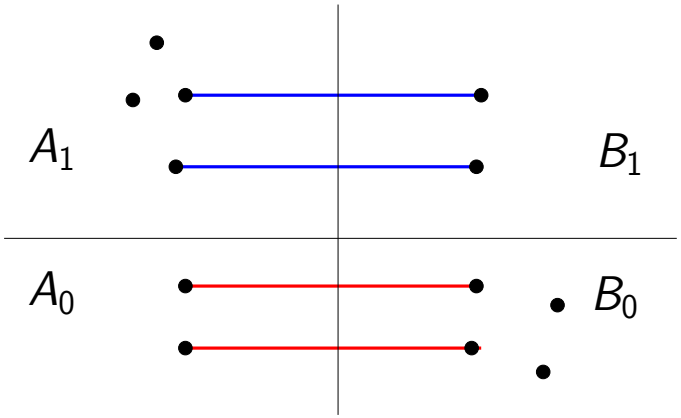
Dual certificate

A useful picture:



So vertices matched along red edges are in $A_0 \cup B_0$.

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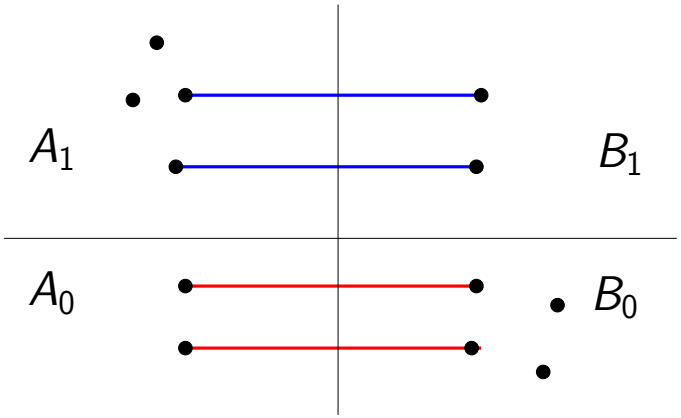

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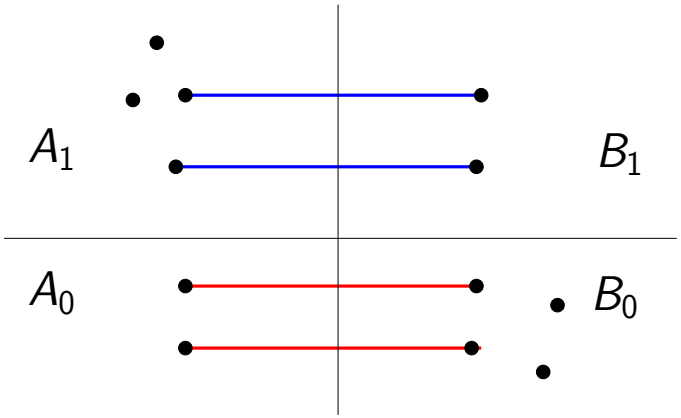

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α-values were assigned as follows:

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Dual feasibility of $\vec{\alpha}$

We need to show this vector $\vec{\alpha}$ is a feasible solution to the dual LP.

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We will also show that $\sum_{u \in A \cup B} \alpha_u = 0$.

- This will mean the dual optimal solution is at most 0.
- This will prove M is a popular matching.

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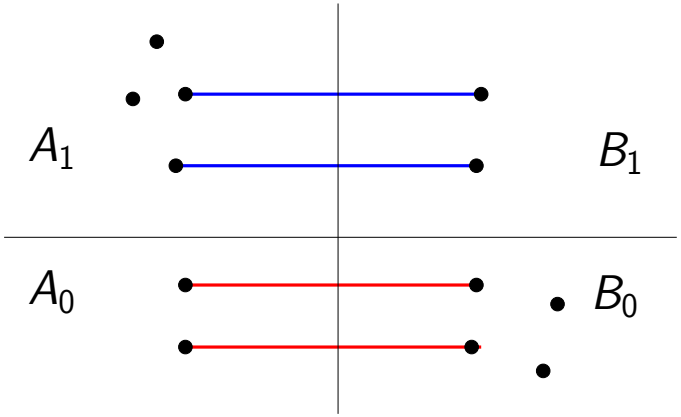
Recall that $\alpha_u \in \{0, \pm 1\}$:



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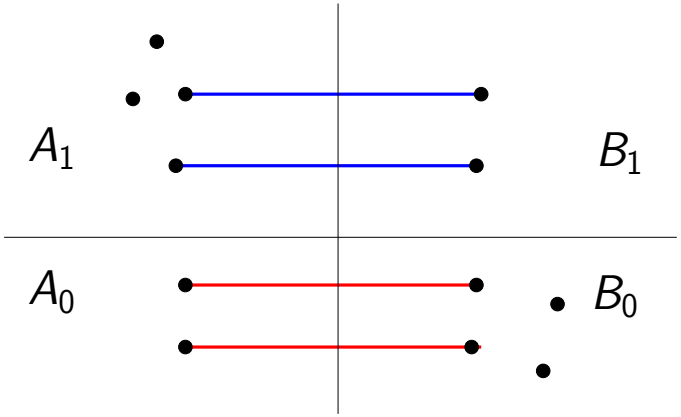

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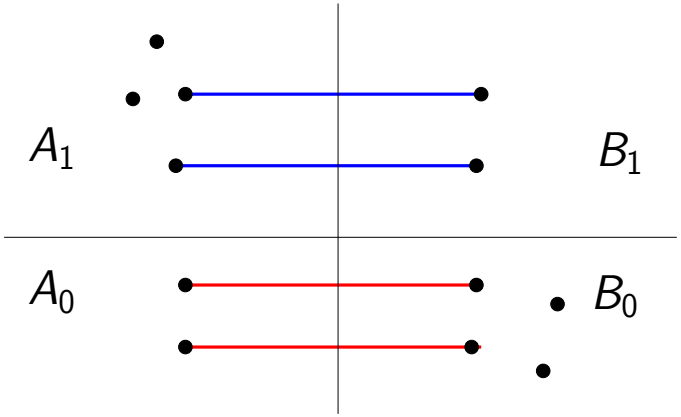

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- We will use the stability of M' in the instance G' to prove the lemma.

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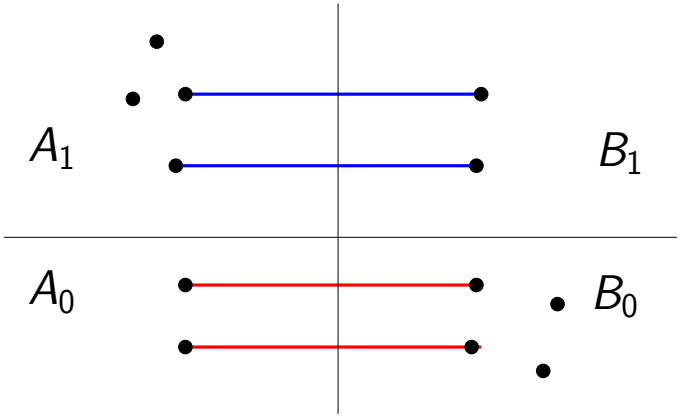

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Lemma. *The constraint $\alpha_a + \alpha_b \geq \mathrm{wt}_M(ab)$ holds for all $ab \in E$.*

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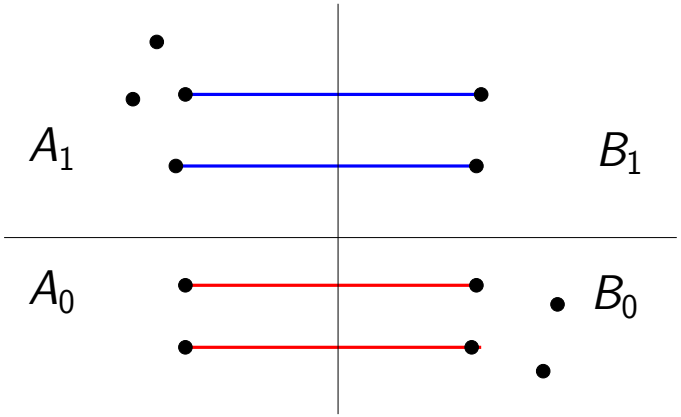
Conclusion. So $\vec{\alpha}$ is dual-feasible.

Optimal value of the dual LP



Every edge in M' is a red edge or a blue edge.

- So $\alpha_a + \alpha_b = 0$ for all $ab \in M$.
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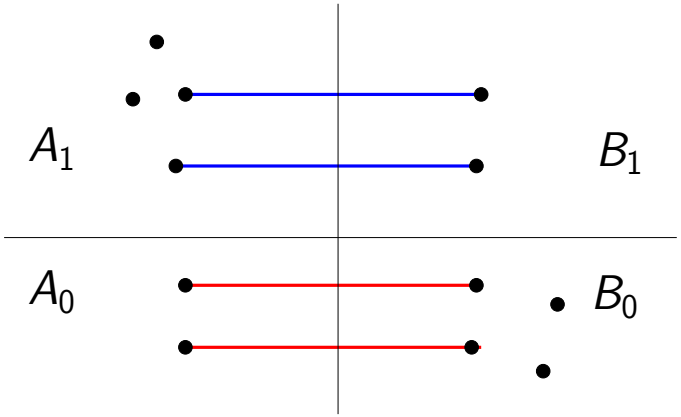

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Thus the optimal value of the dual LP is at most 0.

- Hence M is a popular matching.

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To show $\alpha_a + \alpha_b \geq \mathrm{wt}_M(ab)$ holds for all $ab \in E$.

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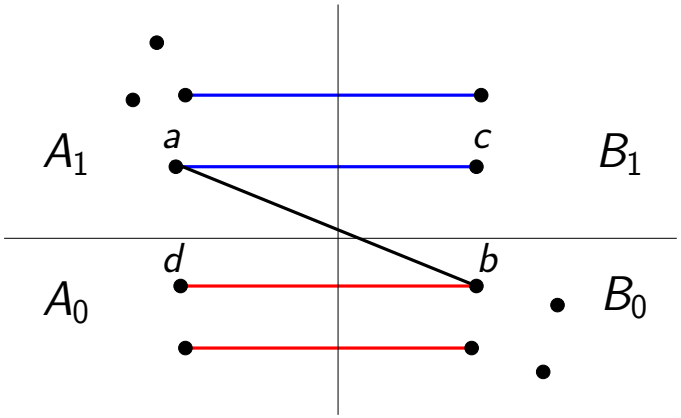

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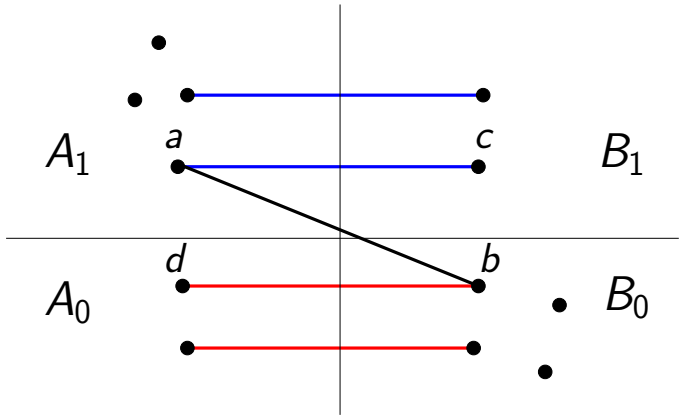

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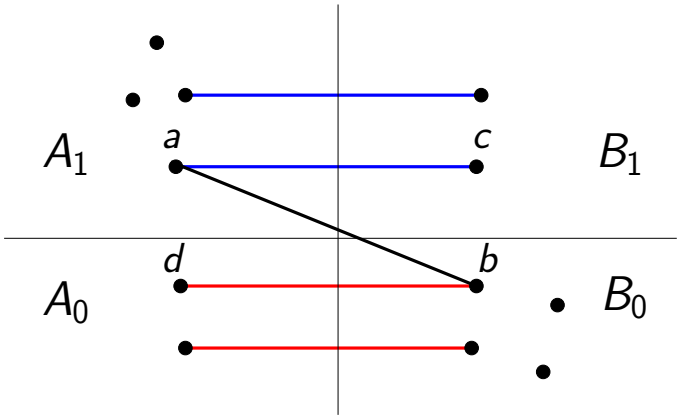

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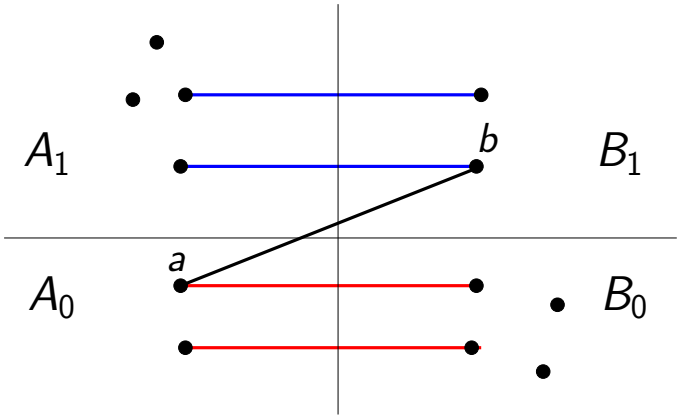
Thus $\text{wt}_M(ab) = -2$, hence $\alpha_a + \alpha_b = -2 = \text{wt}_M(ab)$.

Proof of the lemma

Case 2. Suppose $\alpha_a = \alpha_b = 1$.



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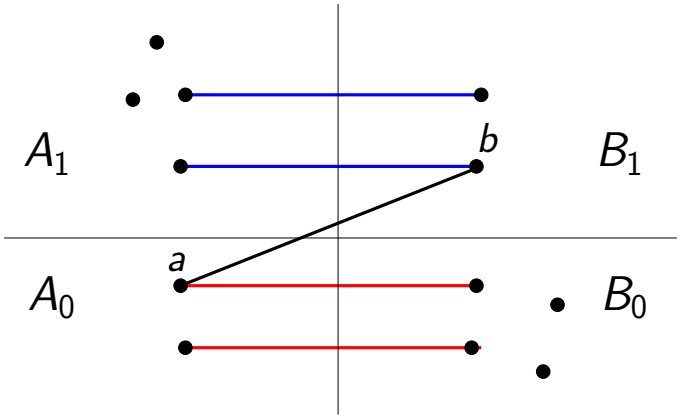

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Case 3. Suppose $\alpha_a = 1$ and $\alpha_b = -1$.

- This means ac and bd are in M' for some neighbors c and d.
- M' is stable in $G' \Rightarrow ab$ does not block M'.

Proof of the lemma

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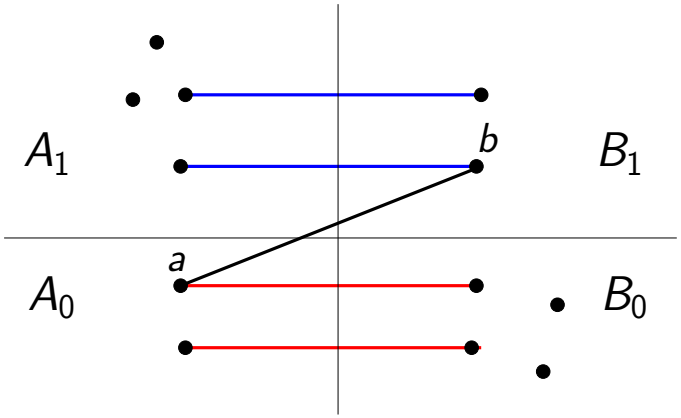

- Since $\text{wt}_M(ab) \in \{0, \pm 2\}$, we have $\alpha_a + \alpha_b = 2 \geq \text{wt}_M(ab)$.

Case 3. Suppose $\alpha_a = 1$ and $\alpha_b = -1$.

- This means ac and bd are in M' for some neighbors c and d.
- M' is stable in $G' \Rightarrow ab$ does not block M'.

Thus $\text{wt}_M(ab) \leq 0$, hence $\alpha_a + \alpha_b = 0 \geq \text{wt}_M(ab)$.

Proof of the lemma

Case 4. Suppose $\alpha_a = -1$ and $\alpha_b = 1$.

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Case 5. Suppose $\alpha_a = 0$.

Since M' is stable in G', ab does not block M'.

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An analogous analysis holds when $\alpha_b = 0$.

Slack edges

For any edge ab incident to an unmatched vertex a:

- Recall that $\alpha_a = 0$ while $\alpha_b = 1$;
- so $\alpha_a + \alpha_b = 1 > 0 = \text{wt}_M(ab)$, thus the edge ab is *slack*.

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Any matching N with a slack edge is not an optimal solution to the primal LP;

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COMPLEMENTARY SLACKNESS

Any matching N with a slack edge is not an optimal solution to the primal LP;

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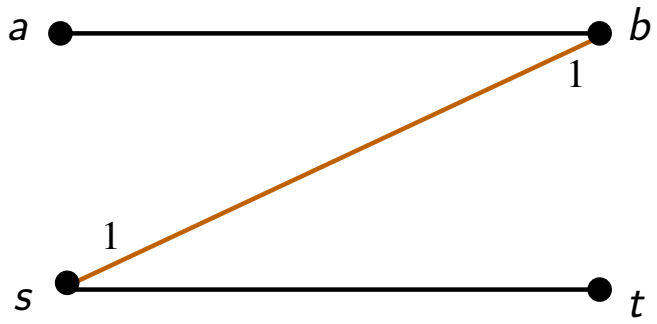
Thus any matching larger than M is unpopular.

- So M is a max-size popular matching. □

Lower bound on $|M|$

Claim. There is no length 3 augmenting path wrt M in G.



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  (a contradiction to M''s stability in G')

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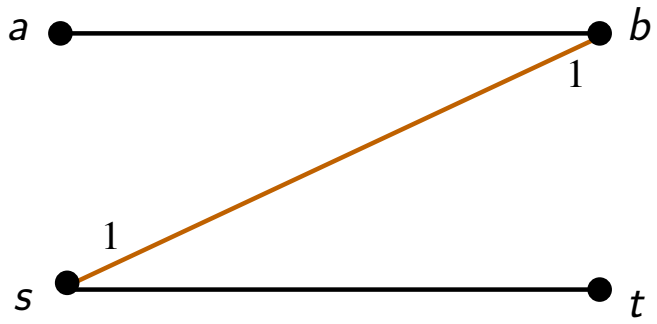

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Hence any augmenting path in $M \oplus M_{\max}$ has length ≥ 5.

- Thus $|M| \geq \frac{2}{3} \cdot |M_{\max}|$.

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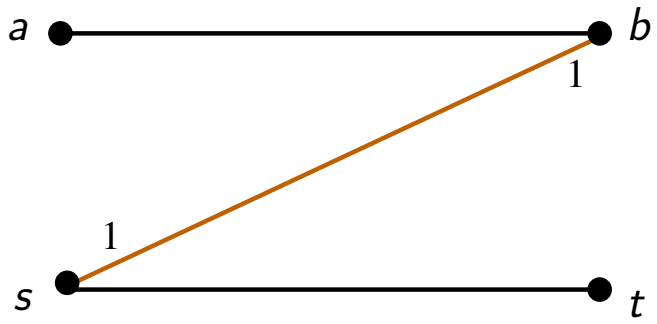

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Thank you!