Popular Matchings - Part II

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Maximum matchings

Applications where the size of the matching is more important than vertex preferences:

- matching medical students to hospitals for residency;
- matching doctors to hospitals in a pandemic;
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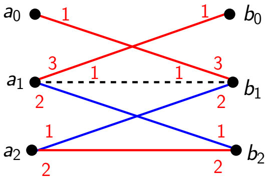
- How about a maximum matching that is popular?

Maximum matchings and popularity

It can be the case that no maximum matching is popular.



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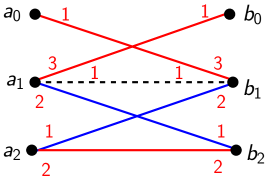

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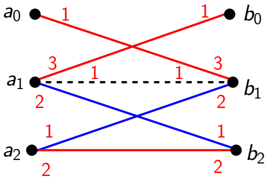

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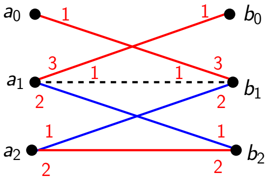

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Does such a "popular maximum matching" always exist in G?

- Furthermore, is it easy to find one?

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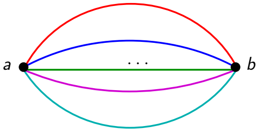
- Every edge ab in G has n parallel copies in G^*: $ab, ab, \ldots, ab, \ldots, ab, ab$.



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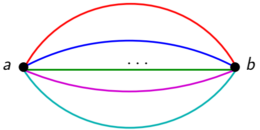

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$$red \succ_a blue \succ_a \cdots \succ_a green \succ_a \cdots \succ_a magenta \succ_a cyan.$$

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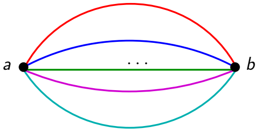

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For any vertex b in B:

$$\textit{cyan} \succ_b \textit{magenta} \succ_b \cdots \succ_b \textit{green} \succ_b \cdots \succ_b \textit{blue} \succ_b \textit{red}.$$

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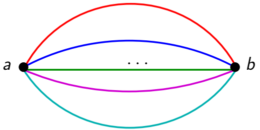

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Within any colour class, every vertex u maintains its original preference order $\succ_u$.

Popular maximum matchings

An extension of our earlier algorithm

- Construct the colourful graph $G^* = (A \cup B, E^*)$.
- Run Gale-Shapley algorithm in G^* to compute M^*.
- Return the corresponding matching M in G.

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Claims 1 and 2 $\Rightarrow$ M is a popular maximum matching.

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- CLAIM 2. $M \succeq M'$ for all maximum matchings M' in G.

Claims 1 and 2 $\Rightarrow$ M is a popular maximum matching.

- Moreover, such a matching can be computed easily.

The LP method

Recall the following edge weight function wt_M in G. For any edge ab:

$$\mathsf{wt}_M(ab) \;=\; \mathsf{vote}_a(b, M(a)) \;+\; \mathsf{vote}_b(a, M(b)).$$

$$\text{Here } \mathsf{vote}_v(u, u') \;=\; \begin{cases} 1 & \text{if } v \text{ prefers } u \text{ to } u' \\ -1 & \text{if } v \text{ prefers } u' \text{ to } u \\ 0 & \text{otherwise.} \end{cases}$$

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So $\mathsf{wt}_M(e) \in \{0, \pm 2\}$ for any edge e.

- Let M be a maximum matching in G.
- OBSERVATION. $\mathsf{wt}_M(N) \leq 0$ for all maximum matchings N
  $\Rightarrow M$ is a popular maximum matching in G.

The LP method

LP for max-weight maximum matching in G:

$$\max \sum_e \mathrm{wt}_M(e) \cdot x_e$$

$$\sum_{e \in \delta(u)} x_e \quad \leq \quad 1 \quad \forall u \in A \cup B$$

$$\sum_{a \in A} \sum_{e \in \delta(a)} x_e \quad = \quad k \quad \text{and} \quad x_e \geq 0 \quad \forall e \in E.$$

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The dual LP

$$\min \quad k \cdot z \;+\; \sum_{u} \alpha_u$$

$$\begin{aligned} \alpha_a + \alpha_b + z &\geq \mathrm{wt}_M(ab) \quad \forall\ ab \in E \\ \alpha_u &\geq 0 \qquad\qquad\ \ \forall\ u \in A \cup B. \end{aligned}$$

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- Recall the colourful graph G^*:
  - let colour 0, colour 1, ..., colour $n-1$ denote the n colours.

A partition of the vertex set $A \cup B$

For $0 \leq i \leq n-1$, let $A_i = \{a \in A : a$ is matched along a colour i edge in $M^*\}$.

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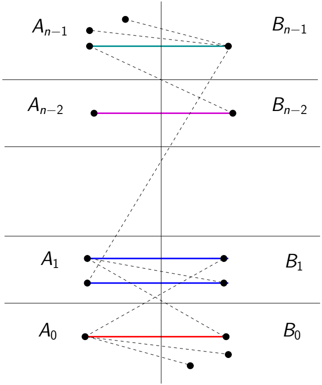

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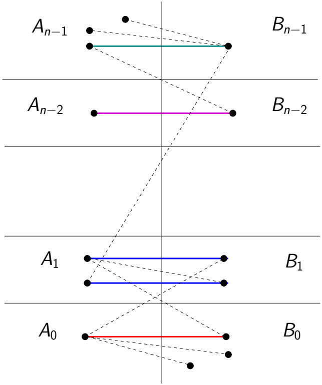

Unmatched vertices of A are in A_{n-1} and unmatched vertices of B are in B_0.

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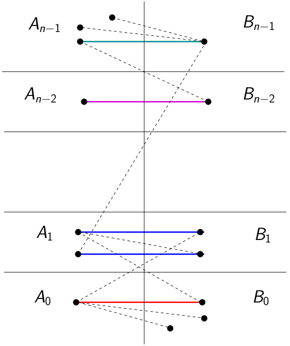

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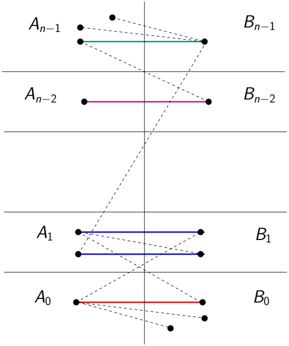

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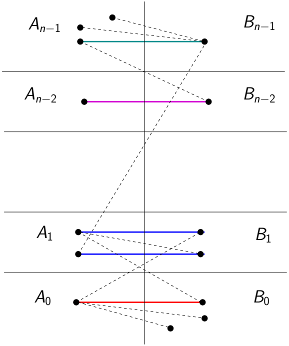

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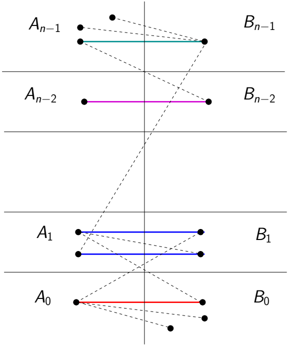

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(4) There is no augmenting path with respect to M.

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Observe that $\alpha_a + \alpha_b + z \;=\; 2(n-1) - 2i + 2i - 2(n-1) \;=\; 0$ for each $ab \in M$.
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So $\vec{\alpha}$ is dual-feasible $\Rightarrow$ the optimal value of the dual LP is at most 0.

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Recall our assignment of α-values for $0 \leq i \leq n-1$:

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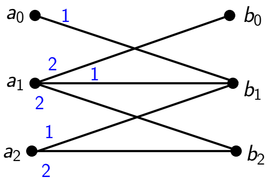
These properties imply that $\alpha_a + \alpha_b + z \geq \text{wt}_M(ab)$ for all $ab \in E$.

- Thus $(\vec{\alpha}, z)$ is dual feasible.
- Hence M is a popular maximum matching.

The original model

Popular matching algorithms were first studied in the model of *one-sided* preferences.



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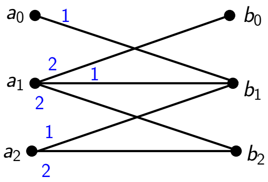

Vertices on the left are agents and those on the right are items.

- Agents have preferences over their neighbours;
- items have no preferences.

This is also called a *house allocation* instance.

Popular matchings in this model

We say $M \succ N$, i.e., M is *more popular* than N, if

$$\underbrace{|\{a \in A : M \succ_a N\}|}_{\#\text{ of votes for } M} > \underbrace{|\{a \in A : N \succ_a M\}|}_{\#\text{ of votes for } N}.$$

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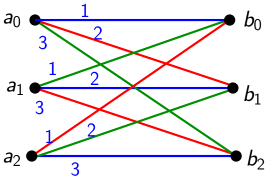

Let us hold elections between some pairs of matchings here.

- Say, between the green matching and the blue matching.

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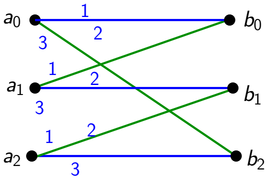

The green matching is more popular than the blue matching.

- In the green vs blue election: green gets 2 votes and blue gets only 1.

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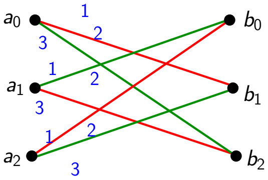

The red matching is more popular than the green matching.

- In the red vs green election: red gets 2 votes and green gets only 1.

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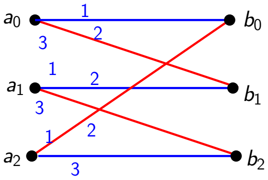

The blue matching is more popular than the red matching.

- In the blue vs red election: blue gets 2 votes and red gets only 1.

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So we have blue $\succ$ red $\succ$ green $\succ$ blue.

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- So this instance has no popular matching.

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- This algorithm works even when there are ties in preferences.
- However it does not work for *partial order* preferences (which are more general).
- Moreover, it does not extend to solve the popular maximum matching problem.

Popular assignments

OUR PROBLEM. Find a popular maximum matching in G, if one exists.

By adding appropriate dummy agents and artificial items to G:

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- *Given an instance $G = (A \cup B, E)$, does G admit a popular assignment?*

- This is a generalization of the popular matching problem.
  - Moreover, agents have partial order preferences.
- Our algorithm is combinatorial and it is based on the LP-method.

The LP-method

Given an assignment M, define edge weights in G as follows. For any edge ab:

$$\mathrm{wt}_M(ab) = \begin{cases} 1 & \text{if } a \text{ prefers } b \text{ to } M(a); \\ -1 & \text{if } a \text{ prefers } M(a) \text{ to } b; \\ 0 & \text{otherwise.} \end{cases}$$

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- M is a popular assignment $\iff$ $\mathrm{wt}_M(N) \le 0$ for all assignments N in G.

The LP-method

LP for max-weight assignment:

$$\max \sum_{e \in E} \mathsf{wt}_M(e) \cdot x_e$$

$$\sum_{e \in \delta(v)} x_e \quad = \quad 1 \;\; \forall v \in A \cup B \quad \text{and} \quad x_e \;\geq\; 0 \;\; \forall e \in E.$$

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CLAIM. M is popular $\iff$ $\exists$ dual feasible $\vec{\alpha}$ such that $\sum_u \alpha_u = 0$ and

- $\alpha_a \in \{0, 1, 2, \ldots, n-1\}$ for all $a \in A$;
- $\alpha_b \in \{0, -1, -2, \ldots, -(n-1)\}$ for all $b \in B$.

Such an $\vec{\alpha}$ will certify that M is a popular assignment in G.

A colour function

Let $c : B \to \{0, 1, 2, \ldots, n-1\}$.

- We can define a subgraph $G_c = (A \cup B, E_c)$ of G as follows.
- For each $a \in A$: let $c^*(a) = \underbrace{\max\{c(b) : b \in \text{Nbr}(a)\}}_{\text{best colour among its neighbours}}$.

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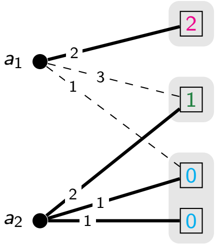
The edge set E_c of G_c has those edges ab where:

- $c(b) = c^*(a)$ and b is a most preferred neighbour for a in colour $c^*(a)$.
- $c(b) = c^*(a) - 1$ and b is a most preferred neighbour for a in the "second best" colour $c^*(a) - 1$ and a prefers b to all neighbours in colour $c^*(a)$.

The subgraph G_c



Here $c^*(a_1) = 2$ and $c^*(a_2) = 1$.

- a_1 does not prefer its neighbour in colour 1 to its neighbour in colour 2;
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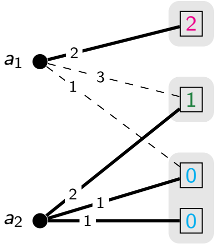

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G_c has a perfect matching $M \Rightarrow M$ is a popular assignment in G.

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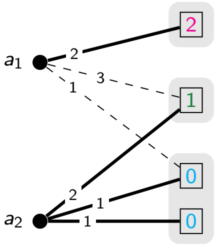

G has a popular assignment if and only if

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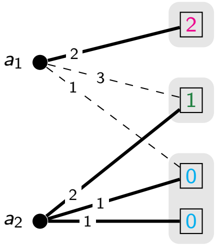

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OUR PROBLEM: Find such a colouring $c : B \to \{0, 1, 2, \ldots, n-1\}$, if there exists one.

Finding a popular assignment/the right c [KKMSS 2022]

1. Initialize $c(b) = 0$ for every $b \in B$.
2. Compute a maximum matching M in the subgraph G_c.
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- In the former case M is a popular assignment with $\vec{\alpha}$ as a dual certificate:

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For the popular matching problem, we will replace "$c(b) = n$" with "$c(b) = 2$".

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Let c_0 be the current c-function in our algorithm.

- Assume $c'(b) \geq c_0(b)$ for all b since $c(b) = 0$ is the initial c-function.

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LEMMA. If b is unmatched in the matching M in G_{c_0} then $c'(b) > c_0(b)$.

When the algorithm says G has no popular assignment

We increase $c_0(b)$ only if b is unmatched in M.

- So for any vertex b unmatched in M: $c'(b) \geq c_0(b) + 1 = c_{\text{new}}(b)$.

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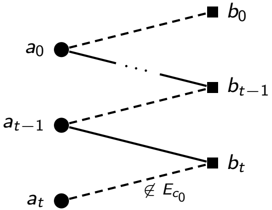
  - $M \oplus M'$ contains an M-augmenting path P in G that starts at b_0.
  - M is a maximum matching in $G_{c_0} \Rightarrow P$ has an edge not in E_{c_0}.
  - Let $b_0 - a_0 - \ldots - b_t - a_t$ be any prefix of P such that $a_t b_t \notin E_{c_0}$.

A proof sketch

We have $a_i b_i \in M' \subseteq E_{c'}$ for all $i \in \{0, \ldots, t\}$.



And $a_i b_{i+1} \in M \subseteq E_{c_0}$ for all $i \in \{0, \ldots, t-1\}$.

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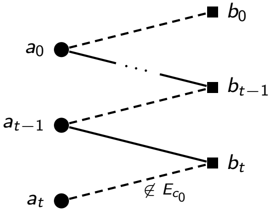

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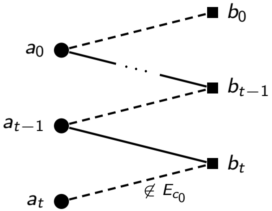
We will first show that $c'(b_t) > c_0(b_t)$.

- This will lead to the proof that $c'(b_0) > c_0(b_0)$.

To show $c'(b_t) > c_0(b_t)$

Since $a_t b_t \notin E_{c_0}$, one of cases 1-3 holds:



1. a_t has a neighbor with colour at least $c_0(b_t) + 2$.
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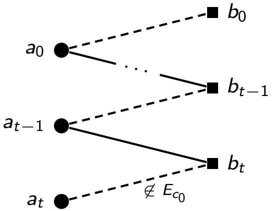

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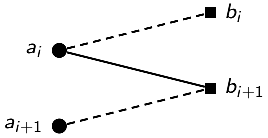
Suppose $c'(b_t) = c_0(b_t)$. Since $c'(b) \geq c_0(b)$ for all b:

- in each case we get $a_t b_t \notin E_{c'}$, a contradiction (since $a_t b_t \in M' \subseteq E_{c'}$).

Thus $c'(b_t) > c_0(b_t)$.

To show that $c'(b_0) > c_0(b_0)$

Suppose $\exists i \in \{0, \ldots, t-1\}$ with $c'(b_{i+1}) > c_0(b_{i+1})$ but $c'(b_i) = c_0(b_i)$.



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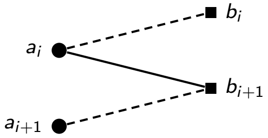


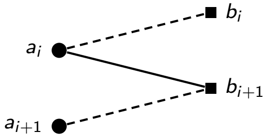
Since $a_i b_{i+1} \in M \subseteq E_{c_0}$, we have the following possibilities:

- $c_0(b_{i+1}) \geq c_0(b_i) + 1$: then $c'(b_{i+1}) \geq c'(b_i) + 2$.
- $c_0(b_{i+1}) = c_0(b_i)$: then $b_i \not\succ_{a_i} b_{i+1}$, but $c'(b_{i+1}) \geq c'(b_i) + 1$.
- $c_0(b_{i+1}) = c_0(b_i) - 1$: then $b_{i+1} \succ_{a_i} b_i$, but $c'(b_{i+1}) \geq c'(b_i)$.

In each case, we get $a_i b_i \notin E_{c'}$ by the definition of $G_{c'}$, a contradiction.

To show that $c'(b_0) > c_0(b_0)$

Suppose $\exists i \in \{0, \ldots, t-1\}$ with $c'(b_{i+1}) > c_0(b_{i+1})$ but $c'(b_i) = c_0(b_i)$.



Since $a_i b_{i+1} \in M \subseteq E_{c_0}$, we have the following possibilities:

- $c_0(b_{i+1}) \geq c_0(b_i) + 1$: then $c'(b_{i+1}) \geq c'(b_i) + 2$.
- $c_0(b_{i+1}) = c_0(b_i)$: then $b_i \not\succ_{a_i} b_{i+1}$, but $c'(b_{i+1}) \geq c'(b_i) + 1$.
- $c_0(b_{i+1}) = c_0(b_i) - 1$: then $b_{i+1} \succ_{a_i} b_i$, but $c'(b_{i+1}) \geq c'(b_i)$.

In each case, we get $a_i b_i \notin E_{c'}$ by the definition of $G_{c'}$, a contradiction.

- Thus $c'(b_i) > c_0(b_i)$ for all $i \in \{0, \ldots, t-1\}$, so $c'(b_0) > c_0(b_0)$. $\square$

Matroids

A matroid $M = (E, \mathcal{I})$:

- E is the ground set;
- $\mathcal{I} \subseteq 2^E$ is called the family of *independent sets*.

The collection of sets $\mathcal{I}$ must satisfy these two axioms:

- $X \subseteq Y$ and $Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$.
- $X, Y \in \mathcal{I}$ and $|Y| > |X| \Rightarrow \exists e \in Y \setminus X$ s.t. $X \cup \{e\} \in \mathcal{I}$.

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An example: partition matroid.

- E is partitioned into disjoint sets $E_1, E_2, \ldots, E_n$;
- $\mathcal{I} = \{X \subseteq E : |X \cap E_i| \leq 1 \ \forall i = 1, \ldots, n\}$.

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Matroid intersection: 2 matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$.

- Independent sets common to both matroids: sets in $\mathcal{I}_1 \cap \mathcal{I}_2$.

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In the intersection of two partition matroids:

(where one matroid is $E = \cup_a \delta(a)$ and each a has preferences over $\delta(a)$)

- a popular matching is a popular common independent set;
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When one matroid is a partition matroid and the other is any arbitrary matroid $(E, \mathcal{I})$:

- the popular assignment algorithm can be generalized to solve:
  - the popular common base/independent set problems [KMSY24];
  - thus several other "popular problems" can be solved efficiently.

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——————————— *Thank you!* ———————————